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Surface Waves in Channels of Finite Depth

Kewal Krishan Puri

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Kewal Krishan Puri

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ABSTRACT

The purpose of the present work is to study surface waves in rectangular channels of finite depth. In all cases we consider two dimensional flow of an incompressible and nonviscous fluid. We also suppose that there is surface tension present in it.

In the first chapter we discuss the feasibility of a steady flow consistent with the linearized problem, and study the corresponding radiation conditions. Such a flow is found to be possible in both the supercritical and the subcritical cases but not in the critical case. In the supercritical case the disturbance dies both upstream and downstream; in the subcritical case, in general, it does not vanish downstream.

The second chapter deals with a study of solitary and periodic waves which result when a stream of uniform flow is disturbed by an arbitrary variable pressure on its free surface. We obtain a general differential equation of the motion. Explicit wave solutions are obtained by considering two special pressure distributions.

In the final chapter we study the progressive wave solutions of the zero order shallow water equations. In the subcritical case, we obtain a wave with a single hump and a single hollow on the two sides of the point of disturbance. In the critical and the supercritical cases, we obtain periodic waves.

Introduction

The purpose of this paper is to study surface waves in an infinite channel of finite depth. The channel in all cases is assumed to have a horizontal bottom and parallel walls.

The phenomenon of surface waves is approached through two approximations, i.e. the amplitude of the wave is small which yields a linear theory or that the depth of the water is small compared to some other significant length such as the radius of curvature of the water surface which results into a non-linear theory. In what follows, the first chapter is based on the former and the remaining two chapters on the latter.

In the first chapter we consider the feasibility of the propagation of disturbance as steady waves. The discussion is carried out within the framework of linear theory. It is assumed that surface tension is present in the water. In cases where such waves exist we seek the corresponding radiation conditions.

This question becomes relevant because as Stoker [9] points out, the steady state formulations are not natural in Newtonian mechanics. The unnaturalness stems from the fact that whereas Newtonian mechanics deals with the initial value problems, the steady state problems are those in which time does not occur at all. The assumption underlying the latter formulations is that after a long period of time, the

propagation of disturbance tends to be steady. The position taken in this chapter is that it is not a priori obvious and therefore it should be investigated. Indeed, our results reveal that when $U^2 = gh$, the steady state does not exist. It exists only for the case $U^2 > gh$ and $U^2 < gh$.

This discussion gives rise to the question as to how far the steady state problems produce a reasonable solution. It is a matter of common experience that such formulations in an infinite domain require more stringent conditions than those in the corresponding time dependent problems. These are the so-called radiation conditions. Aside from an intelligent guess, there is no obvious way to find these except by solving the problem itself. In the present investigations these have been found by actually solving the initial value problem and passing to the limit as $t \longrightarrow \infty$. We find that the disturbance must die out upstream and downstream in case $U^2 > gh$. When $U^2 < gh$, it vanishes upstream but, in general, not downstream.

These ideas were originated by Stoker and he solved the above problem [9]. The present is a generalization of the same when water admits surface tension.

The second approximation as alluded to above develops the classical long wave theory. As propounded by Lamb [4] it has its basis in hydrostatic pressure law. In its modern version, it was obtained by Friedrichs in an appendix to a paper by Stoker [2], as lowest order approximation in the

shallow wave theory. We follow a similar procedure to discuss solitary and cnoidal waves. This discussion forms the contents of the second chapter.

A solitary wave is a wave having a symmetric form with a single hump and which propagates at uniform velocity without change of form. Cnoidal waves are periodic waves. The solitary wave was first observed by Russell in 1844 [5]. The mathematical discussion of it, i.e. its form and the velocity at which it propagates, was initiated by Boussinesq [1]. He was followed by Rayleigh and several other authors. In its next stage of development, Kortweg and DeVries discussed the general class of waves which propagate without change of form. Their investigations produced both the solitary wave and the cnoidal wave.

The basis of the development here is the shallow water theory as given in Stoker [2]. (See also Stoker [6].) The method essentially consists in making a stretching transformation of the variables occurring in the problem, assuming asymptotic expansions of the unknowns in terms of a small parameter and solving for the functions appearing in these expansions. Keller used this technique to discuss solitary and cnoidal waves [7]. A variant of it was used by Stoker and Peters to discuss solitary waves in a medium of varying density [8]. Since then it has been studied in numerous papers. A simplification introduced in all these cases is that the pressure on the free surface is taken to be zero.

Instead, we consider here that an arbitrary pressure is applied at the surface. The result is a differential equation of Kortweg and DeVries type with a variable pressure term in it. Explicit solutions of this equation are obtained in terms of solitary and periodic waves by considering two particular pressure distributions.

The shallow water equations in the lowest order of approximation are known to have the uniform flow as the only smooth solution. In the final chapter, we discuss the nontrivial progressive wave solutions when surface tension is present in water. The discussion involves a parameter $(u_0-a)/\sqrt{gh}$ depending on the original unperturbed flow and the depth of the channel. When this parameter is less than one, we get a wave starting from the point of disturbance and spreading over the whole right half and left half axis with respectively a single hump and a single hollow. In the critical as well as in the supercritical case, i.e. $(u_0-a)^2/gh \geq 1$, the wave becomes a periodic one.

Chapter I

Linear Theory of Steady Waves in Channels of Finite Depth

Formulation of the problem

We consider two dimensional flow of an inviscid incompressible liquid in an infinitely long channel of finite depth, h. The liquid has surface tension T. We suppose that it is initially moving with velocity U parallel to the sides of the channel until some disturbance is created at the free surface. The problem is to study the feasibility of the propagation of this disturbance as a steady wave.

In order to obtain a reasonable solution of a problem posed in an infinite range, it is necessary to impose radiation conditions at $\pm\infty$. Normally, in a physical formulation, boundedness conditions of some kind should suffice. It is common experience, on the other hand, that in time-independent cases, more stringent conditions are required. As to what these conditions should be is not a priori obvious. We seek to find these radiation conditions for the problem mentioned above in the event the steady state does exist.

Such an investigation has already been carried out by Stoker [9] for the case when the fluid has the surface tension 0. The present investigation is a generalization to the case of nonzero surface tension. We study the linearized problem.

We choose our x-axis parallel to the sides of the channel in the equilibrium plane, the y-axis is chosen vertically upward. We assume that the disturbance is created by a pressure $\Pi(x,t)$ on the free surface. Since the motion is originally irrotational, it will remain so always. Hence there exists a velocity potential of the motion. We denote by $\underline{\Phi},\phi$ the complete and the disturbed flow respectively. Thus $\underline{\Phi} = \phi + Ux$. Let $\eta(x,t)$ be the free surface. The equations of motion are:

$$\Delta \underline{\Phi}(x,y;t) = 0$$

$$\frac{p_0}{\rho} + \underline{\Phi}_t + g\eta + \frac{1}{2}(\underline{\Phi}_x^2 + \underline{\Phi}_y^2) = \text{const.} \quad \text{at } y = \eta$$

where p_0 is the pressure at the surface of the liquid.

$$\Pi - p_0 = - \frac{T\eta_{xx}}{(1+\eta_x^2)^{3/2}}$$

Eliminating p_0 between the two foregoing equations we have

$$\frac{\Pi}{\rho} + g\eta + \underline{\Phi}_t + \frac{1}{2}(\underline{\Phi}_x^2 + \underline{\Phi}_y^2) + \frac{T}{\rho} \frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} = \text{const.}$$

The kinematical condition at the surface is

$$\eta_t + \underline{\Phi}_x \eta_x - \underline{\Phi}_y = 0 , \qquad \text{at } y = \eta ;$$
$$\underline{\Phi}_y = 0 , \qquad \text{at } y = -h.$$

The assumption of linearization physically implies that the disturbance generated is so small that the terms of order higher than the first are neglected. The linearized equations are

(1) $$\Delta\phi = 0$$

(2) $$\frac{\Pi}{\rho} + g\eta + \phi_t + U\phi_x + \frac{T}{\rho}\eta_{xx} = 0 \qquad \text{at } y = 0$$

where the constant on the right is absorbed in ϕ.

(3) $$\eta_t + U\eta_x = \phi_y \qquad \text{at } y = 0$$

(4) $$\phi_y = 0 \qquad \text{at } y = -h.$$

At time $t = 0$, we assume

$$\phi(x,y,0) = \eta(x,0) = \Pi(x,0) = 0$$

whence from (2), assuming that it holds on the free surface

$$\phi_t(x,y;0) = 0$$

From now on we assume

$$\Pi(x,t) \equiv \Pi(x) \qquad \text{for } t > 0.$$

Finally, the conditions at $\pm\infty$ for ϕ are considered in the existence of the Fourier Transform $\bar{\phi}$ of ϕ and its

derivatives. The conditions on $\Pi(x,t)$ are also prescribed through its Fourier transform and are kept open for the time being.

$$\bar{\phi} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x,y;t)\, e^{-isx}\, dx$$

$$= \frac{1}{i\sqrt{2\pi}\, s} \int_{-\infty}^{\infty} \phi_x(x,y,t)\, e^{-isx}\, dx \ .$$

$$-s^2\bar{\phi} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_{xx}(x,y,t)\, e^{-isx}\, dx.$$

Also

$$\bar{\phi}_{yy} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_{yy}(x,y,t)\, e^{-isx}\, dx.$$

Thus (1) becomes

$$\bar{\phi}_{yy} - s^2\bar{\phi} = 0 \ . \tag{5}$$

Similarly (2), (3) become

$$\frac{\bar{\Pi}}{\rho} + g\bar{\eta} + \bar{\phi}_t + U(i\bar{\phi}s) - s^2 \frac{T}{\rho}\bar{\eta} = 0 \tag{6}$$

$$\bar{\eta}_t + isU\bar{\eta} - \bar{\phi}_y = 0 \tag{7}$$

where $\bar{\eta}, \bar{\Pi}$ are the transforms of η, Π respectively. Solving equation (5), we get

(8) $$\bar{\phi}(s,y;t) = A(s,t)\cosh s\,(y+h)$$

$$\bar{\eta}\left(s^2\frac{T}{\rho} - g\right) = \frac{\bar{\Pi}}{\rho} + \bar{\phi}_t + is\bar{\phi}U$$

$$\bar{\eta}_t\left(s^2\frac{T}{\rho} - g\right) = \bar{\phi}_{tt} + is\bar{\phi}_t U \ .$$

Substituting for $\bar{\eta}, \bar{\eta}_t$ in equation (7) we get

(9) $$\bar{\phi}_{tt} + 2isU\bar{\phi}_t - s^2U^2\bar{\phi} - \bar{\phi}_y\left(s^2\frac{T}{\rho} - g\right) = -isU\frac{\bar{\Pi}}{\rho} \ .$$

Substituting for $\bar{\phi}$ from equation(8), and evaluating at $y = 0$, we have:

$$A_{tt}\cosh sh + 2isUA_t\cosh sh - A\{s^2U^2\cosh sh + s\left(s^2\frac{T}{\rho} - g\right)\sinh sh\} = -isU\frac{\bar{\Pi}}{\rho} \ .$$

$$A_{tt} + 2isUA_t - A\{s^2U^2 + s\left(s^2\frac{T}{\rho} - g\right)\tanh sh\} = -isU\frac{\bar{\Pi}}{\rho}\operatorname{sech} sh.$$

In order to solve the homogeneous equation, we have to solve the quadratic:

$$D^2 + 2isUD - \{s^2U^2 + s\left(s^2\frac{T}{\rho} - g\right)\tanh sh\} = 0 \ .$$

$$D = -isU \pm \{s\left(s^2\frac{T}{\rho} - g\right)\tanh sh\}^{1/2} \ .$$

The solution of the equation, therefore, is

$$A(s,t) = \frac{isU\overline{\prod}}{\rho \cosh sh}\left\{\frac{1}{s^2U^2 - s(g - s^2\frac{T}{\rho})\tanh sh}\right\}$$

$$+ B\, e^{-it(sU + \sqrt{s(g - s^2\frac{T}{\rho})\tanh sh})}$$

$$+ C\, e^{-it(sU - \sqrt{s(g - s^2\frac{T}{\rho})\tanh sh})}, \quad \text{for } |s| \leq \sqrt{\frac{\rho g}{T}};$$

and

$$A(s,t) = \frac{isU\overline{\prod}}{\rho \cosh sh}\left\{\frac{1}{s^2U^2 + s(s^2\frac{T}{\rho} - g)\tanh sh}\right\}$$

$$+ B_1\, e^{(-isU + \sqrt{s(s^2\frac{T}{\rho} - g)\tanh sh})t}$$

$$+ C_1\, e^{(-isU - \sqrt{s(s^2\frac{T}{\rho} - g)\tanh sh})t}, \quad \text{for } |s| > \sqrt{\frac{\rho g}{T}}.$$

B, C, B_1, C_1 are constants of integration to be evaluated from the initial conditions $A(s,0) = 0$ and $A_t(s,0) = 0$. These follow from equation (8) and the fact that $\overline{\phi}(s,0) = \overline{\phi}_t(s,0) = 0$. Thus

$$B + C = -\frac{isU\overline{\prod}}{\rho \cosh sh}\left\{\frac{1}{s^2U^2 - s(g - s^2\frac{T}{\rho})\tanh sh}\right\}.$$

$$B(sU + \sqrt{s(g - s^2\frac{T}{\rho})\tanh sh}) + C(sU - \sqrt{s(g - s^2\frac{T}{\rho})\tanh sh}) = 0.$$

$$C = \frac{-isU\overline{\prod}}{2\rho \cosh sh \sqrt{s(g - s^2\frac{T}{\rho})\tanh sh}}\left\{\frac{1}{sU - \sqrt{s(g - s^2\frac{T}{\rho})\tanh sh}}\right\}.$$

$$B = \frac{isU\overline{\prod}}{2\rho \cosh sh \sqrt{s(g-s^2\frac{T}{\rho})\tanh sh}} \cdot \left\{ \frac{1}{sU+\sqrt{s(g-s^2\frac{T}{\rho})\tanh sh}} \right\} .$$

Similarly,

$$B_1 = \frac{-isU\overline{\prod}}{2\rho \cosh sh \sqrt{s(s^2\frac{T}{\rho}-g)\tanh sh}} \cdot \left\{ \frac{isU+\sqrt{s(s^2\frac{T}{\rho}-g)\tanh sh}}{s^2U^2+s(s^2\frac{T}{\rho}-g)\tanh sh} \right\} .$$

$$C_1 = \frac{-isU\overline{\prod}}{2\rho \cosh sh \sqrt{s(s^2\frac{T}{\rho}-g)\tanh sh}} \cdot \left\{ \frac{-isU+\sqrt{s(s^2\frac{T}{\rho}-g)\tanh sh}}{s^2U^2+s(s^2\frac{T}{\rho}-g)\tanh sh} \right\} .$$

$$A = \begin{cases} \dfrac{isU\sqrt{\pi}}{\rho \cosh sh} \left[\dfrac{1}{s^2U^2 - s(g - s^2\frac{T}{\rho})\tanh sh} \right. \\ \quad + \dfrac{e^{-i\{sU + \sqrt{s(g-s^2\frac{T}{\rho})\tanh sh}\}t}}{2\sqrt{(g-s^2\frac{T}{\rho})s\tanh sh}\{sU + \sqrt{s(g-s^2\frac{T}{\rho})\tanh sh}\}} \\ \quad \left. - \dfrac{e^{-i\{sU - \sqrt{s(g-s^2\frac{T}{\rho})\tanh sh}\}t}}{2\sqrt{(g-s^2\frac{T}{\rho})s\tanh sh}\{sU - \sqrt{s(g-s^2\frac{T}{\rho})\tanh sh}\}} \right] \\ \qquad \text{for } |s| \leq \sqrt{\frac{g\rho}{T}} = a \\ \text{and} \\ \dfrac{isU\sqrt{\pi}}{\rho \cosh sh} \left[\dfrac{1}{s^2U^2 + s(s^2\frac{T}{\rho} - g)\tanh sh} \right. \\ \quad - \dfrac{e^{\left(isU + \sqrt{s(s^2\frac{T}{\rho} - g)\tanh sh}\right)t}}{2\sqrt{s(s^2\frac{T}{\rho} - g)\tanh sh}\{-isU + \sqrt{s(s^2\frac{T}{\rho} - g)\tanh sh}\}} \\ \quad \left. - \dfrac{e^{\left(-isU - \sqrt{s(s^2\frac{T}{\rho} - g)\tanh sh}\right)t}}{2\sqrt{s(s^2\frac{T}{\rho} - g)\tanh sh}\{isU + \sqrt{s(s^2\frac{T}{\rho} - g)\tanh sh}\}} \right] \\ \qquad \text{for } |s| > \sqrt{\frac{\rho g}{T}} \ . \end{cases}$$

We can solve for ϕ by inverse transformation

$$\phi(x,y;t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(s,t) \cosh s(y+h)\, e^{isx}\, ds \ .$$

The convergence of the above integral for a finite t.

$$\phi(x,y,t) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \frac{isU\sqrt{\ }\cosh s\ (y+h)\ e^{isx}}{\rho \cosh sh}$$

$$\cdot\left[\frac{1}{s^2U^2 - s(g-s^2\frac{T}{\rho})\tanh sh} + \frac{1}{2\sqrt{s(g-s^2\frac{T}{\rho})\tanh sh}}\right.$$

$$\left.\cdot\left\{\frac{e^{-i\{sU+\sqrt{s(g-s^2\frac{T}{\rho})\tanh sh}\}t}}{sU+\sqrt{s(g-s^2\frac{T}{\rho})\tanh sh}} - \frac{e^{-i\{sU-\sqrt{s(g-s^2\frac{T}{\rho})\tanh sh}\}t}}{sU-\sqrt{s(g-s^2\frac{T}{\rho})\tanh sh}}\right\}\right] ds$$

$$+ \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{-a} + \int_{a}^{\infty}\right)\frac{isU\sqrt{\ }\cosh s\ (y+h)\ e^{isx}}{\rho \cosh sh}$$

$$\cdot\left[\frac{1}{s^2U^2 + s(s^2\frac{T}{\rho} - g)\tanh sh} + \frac{1}{2\sqrt{s(s^2\frac{T}{\rho} - g)\tanh sh}}\right] \cdot$$

$$\cdot\left\{-\frac{e^{-isUt+t\sqrt{s(s^2\frac{T}{\rho}-g)\tanh sh}}}{-isU+\sqrt{s(s^2\frac{T}{\rho}-g)\tanh sh}}-\frac{e^{-isUt-t\sqrt{s(s^2\frac{T}{\rho}-g)\tanh sh}}}{isU+\sqrt{s(s^2\frac{T}{\rho}-g)\tanh sh}}\right\}\Bigg]ds$$

$$= \phi_a + \phi_{-\infty} + \phi_{\infty} \qquad \text{where } a^2 = \frac{\rho g}{T},$$

where ϕ_a is the first integral from -a to a and $\phi_{-\infty}$ and ϕ_{∞} are the remaining two integrals in $(-\infty,-a)$ and (a,∞) respectively.

We first consider ϕ_a. The function outside the parenthesis does not have any singularity within the interval of integration. We claim that the function inside the parenthesis is bounded. This can be demonstrated by expanding the exponentials. Thus

$$\frac{1}{s^2U^2-s(g-s^2\frac{T}{\rho})\tanh sh}+\frac{1}{2\sqrt{s(g-s^2\frac{T}{\rho})\tanh sh}}$$

$$\cdot\left\{\frac{e^{-i(sU+\sqrt{s(g-s^2\frac{T}{\rho})\tanh sh})t}}{sU+\sqrt{s(g-s^2\frac{T}{\rho})\tanh sh}}-\frac{e^{-it(sU-\sqrt{s(g-s^2\frac{T}{\rho})\tanh sh}}}{sU-\sqrt{s(g-s^2\frac{T}{\rho})\tanh sh}}\right\}$$

$$= \frac{1}{s^2U^2-s(g-s^2\frac{T}{\rho})\tanh sh}+\frac{1}{2\sqrt{(g-s^2\frac{T}{\rho})\tanh sh}}$$

$$\cdot\Big[(sU-\sqrt{s(g-s^2\frac{T}{\rho})\tanh sh})\ \{1-it(sU+\sqrt{s(g-s^2\frac{T}{\rho})\tanh sh})$$

$$-\frac{1}{2}t^2(sU+\sqrt{s(g-s^2\frac{T}{\rho})\tanh sh})^2+\ldots\}-(sU+\sqrt{s(g-s^2\frac{T}{\rho})\tanh sh})$$

$$\cdot\{1-it(sU-\sqrt{s(g-s^2\frac{T}{\rho})\tanh sh})-\frac{1}{2}t^2(sU-\sqrt{s(g-s^2\frac{T}{\rho})\tanh sh})^2$$

$$+\ldots\}\Big]\frac{1}{s^2U^2-s(g-s^2\frac{T}{\rho})\tanh sh}$$

$$=\frac{1}{s^2U^2-s(g-s^2\frac{T}{\rho})\tanh sh}\left[1-\frac{1}{2\sqrt{s(g-s^2\frac{T}{\rho})\tanh sh}}\cdot\right.$$

$$\left.\cdot\left\{2t\sqrt{s(g-s^2\frac{T}{\rho})\tanh sh}\ (1+t^2(s^2U^2-s(g-s^2\frac{T}{\rho})\tanh sh))\right\}\right]$$

$$=1+\{s^2U^2-s(g-s^2\frac{T}{\rho})\tanh sh\}+\ldots\ .$$

That is, all the singularities of the integrand are cancelled out. Since s takes on only finite values, the function in the parenthesis is bounded. Let its bound be M.

$$|\phi_a|\leq\frac{M}{\sqrt{2\pi}}\int_0^a\frac{sU(\overline{\prod}(s)+\overline{\prod}(-s))\cosh s(y+h)}{\rho\cosh sh}\,ds\ ,$$

which, being finite, implies that ϕ_a converges.

In the case of ϕ_∞, the singularity at the point $s = a$ is a square root singularity and is therefore integrable. On the other hand, for large $|s|$,

$$\frac{s \cosh s\,(y+h)}{\cosh sh} = s\,\frac{e^{s(y+h)} + e^{-s(y+h)}}{e^{sh} + e^{-sh}} \approx s\, e^{y|s|}.$$

The integrand in the absolute value behaves as

$$\frac{sU}{\sqrt{2\pi}}\,|\overline{\overline{\Pi}}(s)|e^{y|s|}\left[\left|\frac{1}{s^2U^2+s(s^2\frac{T}{\rho}-g)\tanh sh}\right|\right.$$

$$\left. + \frac{1}{2\sqrt{s(s^2\frac{T}{\rho}-g)\tanh sh}}\right.$$

$$\left.\cdot\left\{\frac{e^{t\sqrt{s(s^2\frac{T}{\rho}-g)\tanh sh}} + e^{-t\sqrt{s(s^2\frac{T}{\rho}-g)\tanh sh}}}{\sqrt{s^2U^2 + s(s^2\frac{T}{\rho}-g)\tanh sh}}\right|\right.$$

$$\approx |\overline{\overline{\Pi}}(s)|\, se^{y|s|}\, e^{t\,s^{3/2}} \qquad \text{for large } s.$$

The integral ϕ_∞ then will be convergent, provided $\overline{\overline{\Pi}}(s)$ behaves like e^{-s^2} for large values of s. Indeed the subsequent analysis reveals that $\overline{\overline{\Pi}}(s)$ should be of the form $e^{-s^2}g(s)$, where $g(s)$ remains bounded at the two

infinities. Henceforth, therefore, we assume that $\overline{\Pi}(s) \equiv e^{-s^2} g(s)$, where $g(s)$ is analytic for all s and is bounded at $\pm\infty$. The function $g(s)$, for instance, can be a periodic function. The convergence of $\phi_{-\infty}$ can be discussed the same way.

We write

$$\phi_x = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(s,t)\, s \cosh s(y+h) e^{isx}\, ds \ .$$

This integral then gives the velocity of propagation of the disturbance at any finite time t. For the existence of steady waves we have to study ϕ_x after a long time over the whole of the x-axis i.e. $\lim_{t\to\infty} \phi_x$.

$$\lim_{t\to\infty} \phi_x = \frac{i}{\sqrt{2\pi}} \lim_{t\to\infty} \left[\int_{-a}^{a} A(s,t)\, s \cosh s(y+h) e^{isx}\, ds \right.$$

$$\left. + \left(\int_{-\infty}^{-a} + \int_{a}^{\infty} \right) A(s,t)\, s \cosh s\,(y+h) e^{isx}\, ds \right]$$

$$= \frac{i}{\sqrt{2\pi}} \left(I_a + I_{-\infty} + I_{\infty} \right)$$

$$I_a = \lim_{t\to\infty} \int_{-a}^{a} \frac{is^2 U \overline{\Pi} e^{isx}}{\rho \cosh sh} \cosh s(y+h) \left[\frac{1}{s^2U^2 - s\frac{T}{\rho}(a^2-s^2)\tanh sh} \right.$$

$$\left. + \frac{1}{2\sqrt{\frac{T}{\rho} s(a^2-s^2)\tanh sh}} \left\{ \frac{e^{-if_+ t}}{f_+} - \frac{e^{-if_- t}}{f_-} \right\} \right] ds \ ,$$

where

$$f_{\pm} = sU \pm \sqrt{s(g - s^2 \frac{T}{\rho}) \tanh sh}$$

(11)

Let $I_a = I_{as} + I_{at}$, where I_{as} is the time independent part.

We first consider I_{at}.

This requires the investigation of the zeros of f_+, f_-.

We write

$$F(s) = f_+ \cdot f_-$$

$$= s^2U^2 - (g - s^2 \frac{T}{\rho}) s \tanh sh$$

$$= sg\left[\frac{U^2}{gh} hs - (1 - \frac{s^2T}{g\rho}) \tanh sh\right] .$$

$F(s) = 0$, if $s = 0$, or

$$\frac{U^2}{gh} \cdot hs = \left(1 - s^2 \frac{T}{g\rho}\right) \tanh hs \quad .$$

It is obvious that this equation also has a zero at $s = 0$. Hence $F(s)$ has double zero at $s = 0$. For $s \neq 0$, we write

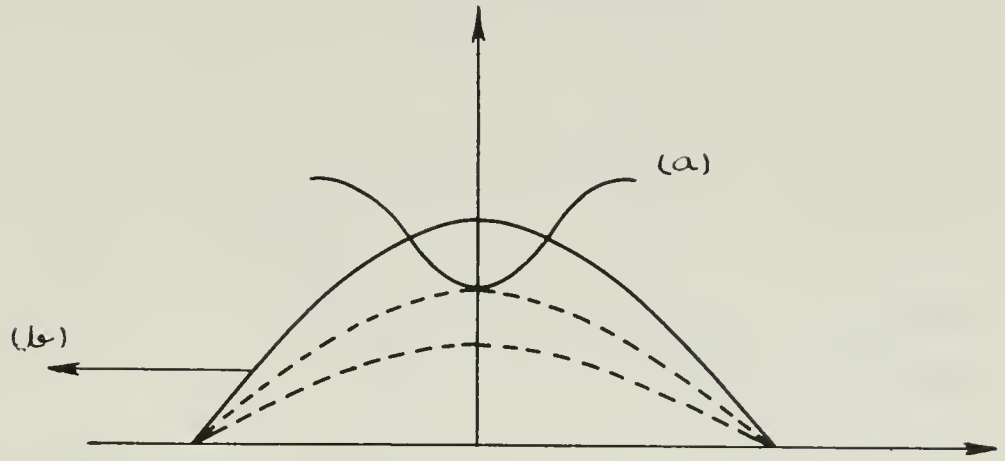
$$\frac{U^2}{gh} hs \coth hs = 1 - s^2h^2 \frac{T}{\rho gh^2}$$

$$= y \frac{U^2}{gh} , \quad \text{say.}$$

Thus the zeros of F(s) other than s = 0 are the intersection of the curves:

$$y = hs \coth hs \qquad (a)$$

$$\frac{U^2}{gh} y = 1 - \frac{T}{\rho g h^2} s^2 h^2 \qquad (b)$$



It follows that for

$\frac{U^2}{gh} > 1$, F(s) has a zero of second order at s = 0

$\frac{U^2}{gh} = 1$, F(s) has a zero of fourth order at s = 0

$\frac{U^2}{gh} < 1$, F(s) has a zero of second order at s = 0

and two simple zeros at $s = \beta, -\beta$; $|\beta| \leq \rho g/T$. We also observe that these functions can be defined as analytic

and single valued in a neighborhood of real axis cut along the negative real axis from -a to 0 and along the positive axis from a to ∞. To start with, we suppose $U^2/gh > 1$,

$$I_{at} = \lim_{t\to\infty} \int_{-a}^{a} \frac{is^2 U\sqrt{\rho T}\, e^{isx} \cosh s(y+h)}{2\sqrt{\rho T}\cosh sh \sqrt{s(a^2-s^2)\tanh sh}} \left(\frac{e^{-if_+ t}}{f_+} - \frac{e^{-if_- t}}{f_-}\right) ds.$$

The path of integration is along the real axis. The singularities of the integrand are at $s = \pm a$. The double zero at $s = 0$ of the denominator of the integrand is neutralized by the factor s^2 in the numerator. We write

$$I_{at} = \lim_{t\to\infty} \left[\left\{ \int_{-a}^{-r_1} + \int_{-r_1}^{r_1} + \int_{r_1}^{a} \right\} \frac{is^2 U\sqrt{\rho T}\, e^{isx} \cosh s(y+h)}{2\sqrt{\rho T}\cosh sh \sqrt{s(a^2-s^2)\tanh sh}} \cdot \left(\frac{e^{-itf_+}}{f_+} - \frac{e^{-itf_-}}{f_-} \right) ds \right]$$

$$= \lim_{t\to\infty} \left\{ I_{-a} + I_{r_1} + I_a \right\}$$

where r_1 is so small that $f'_{\pm}(s)$ do not have any zero other than possibly $s = 0$ in $[-r_1, r_1]$. This implies that $f_{\pm}(s)$ do not have any stationary point in $[-r_1, r_1]$ when $U^2 > gh$. Hence $I_{r_1} = 0(1/t) \to 0$ as $t \to \infty$.

$$I_{-a} = \lim_{t\to\infty} \int_{-a}^{-r_1} \sum_{n=0}^{\infty} b_n (s+a)^{n-\frac{1}{2}} e^{isx} e^{-itf_+} ds$$

$$+ \lim_{t\to\infty} \int_{-a}^{-r_1} \sum_{n=0}^{\infty} c_n (s+a)^{n-\frac{1}{2}} e^{isx} e^{-itf_-} ds$$

$$= \lim_{t\to\infty} \left[\int_{-a}^{-r_1} \frac{b_0}{\sqrt{s+a}} e^{isx} e^{-itf_+} ds \right.$$

$$\left. + \int_{-a}^{-r_1} \frac{c_0}{\sqrt{s+a}} e^{isx} e^{-itf_-} ds \right]$$

$$+ \lim_{t\to\infty} \left[\int_{-a}^{-r_1} \sum_{n=1}^{\infty} b_n (s+a)^{n-\frac{1}{2}} e^{isx} e^{-itf_+} ds \right.$$

$$\left. + \int_{-a}^{-r_1} \sum_{n=1}^{\infty} c_n (s+a)^{n-\frac{1}{2}} e^{isx} e^{-itf_-} ds \right] . \tag{13}$$

$$\lim_{t\to\infty} \int_{-a}^{-r} \sum_{n=1}^{\infty} (s+a)^{n-\frac{1}{2}} e^{isx} e^{-itf_+} ds .$$

The series, being Taylor's expansion is uniformly convergent. Hence

$$\lim_{t\to\infty} \int_{-a}^{-r_1} \sum_{n=1}^{\infty} (s+a)^{n-\frac{1}{2}} e^{isx} e^{-itf_+} ds$$

$$= \lim_{t\to\infty} \sum_{n=1}^{\infty} \int_{-a}^{-r_1} (s+a)^{n-\frac{1}{2}} e^{isx} e^{-itf_+} ds$$

$$(14) \qquad = \sum_{n=1}^{\infty} \left[\lim_{t\to\infty} \int_{-a}^{-r_1} (s+a)^{n-\frac{1}{2}} e^{isx} e^{-itf_+} ds \right].$$

To consider these integrals and, analogously, those in the second parentheses in (13) we have to find the stationary points of $f_{\pm}(s)$.

$$f_+ = sU + \sqrt{s(a^2-s^2)\tanh hs}\,\sqrt{\frac{T}{\rho}}$$

$$f_+' = U + \sqrt{\frac{T}{\rho}} \left\{ \frac{(a^2-3s^2)\sqrt{\tanh hs}}{2\sqrt{sa^2-s^3}} + \frac{h}{2} \frac{\operatorname{sech}^2 hs}{\sqrt{\tanh hs}} \sqrt{sa^2-s^3} \right\}$$

for $s \neq \pm a$.

Clearly, $s = 0$ is not a zero of $f_+'(s)$. On the other hand, $f_+'(s) = 0$ for s satisfying

$$U\sqrt{\tanh hs(sa^2-s^3)} + \frac{1}{2}\sqrt{\frac{T}{\rho}} \left\{ (a^2-3s^2)\tanh hs + h \operatorname{sech}^2 hs\,(sa^2-s^3) \right\} = 0 .$$

We write

$$(15) \qquad y = U\sqrt{\tanh hs\ (sa^2 - s^3)}$$

$$(16) \qquad y = -\frac{1}{2}\sqrt{\frac{T}{\rho}}\ \{(a^2 - 3s^2)\tanh hs + h(sa^2 - s^3)\operatorname{sech}^2 hs\}$$

and look for the intersection of these two curves.
Equation (15) has to be understood as a positive branch of

$$y^2 = U^2 \tanh hs\ (sa^2 - s^3)\ .$$

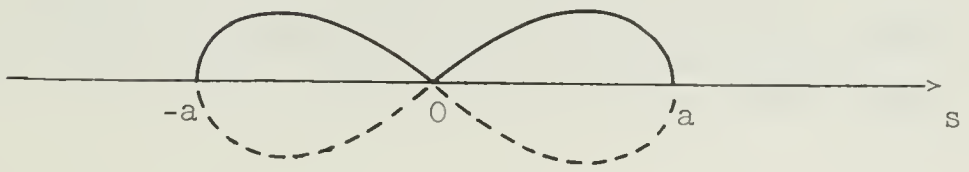
This curve is symmetrical about the y-axis and s-axis.
Also, (0,0) lies on it.

$$2\frac{dy}{ds} = U\frac{\{h\ \operatorname{sech}^2 hs(sa^2 - s^3) + (a^2 - 3s^2)\tanh hs\}}{\sqrt{\tanh hs\ (sa^2 - s^3)}}$$

$\frac{dy}{ds} = \infty$ for $s = \pm a$ only

$\frac{dy}{ds} = 0$ for $h\ \operatorname{sech}^2 hs(sa^2 - s^3) + (a^2 - 3s^2)\tanh hs = 0$,

i.e. for s in $[\frac{a}{\sqrt{3}}, a]$. The curve (15) therefore is represented by the solid line below.



We now consider the curve:

$$y = -\frac{1}{2}\sqrt{\frac{T}{\rho}}\{(a^2 - 3s^2)\tanh hs + h\,\mathrm{sech}^2 hs(sa^2 - s^3)\}$$

$$y(0) = 0 \quad \text{and} \quad y < 0 \quad \text{in} \quad (0, \frac{a}{\sqrt{3}})$$

$$y(a) = \sqrt{\frac{T}{\rho}}\, a^2 \tanh sh > 0 \ ;$$

we know that

$$y(a) = 0 \quad \text{for (15)}.$$

To disgintuish the two curves we write,

$$y_1 = U\sqrt{\tanh hs\,(sa^2 - s^3)}$$

$$y_2 = -\frac{1}{2}\sqrt{\frac{T}{\rho}}\{(a^2 - 3s^2)\tanh hs + h\,\mathrm{sech}^2 hs(sa^2 - s^3)\};$$

then

$$\frac{dy_1}{ds} = -\sqrt{\frac{\rho}{T}} \cdot \frac{U^2}{y_1} \cdot y_2 \ .$$

Thus

$$\frac{dy_1}{ds} = 0 \quad \text{implies} \quad y_2 = 0 \ .$$

Since $dy_1/ds = 0$ for $s < a$, it follows that the curve (16) must enter the loop of (15). Also since we know that $y_2(a) > 0$, (15) and (16) must intersect. Hence $f_+'(s)$ must have at least one zero. Because of $f_+'(s)$ being analytic in the open interval (-a,a), there

can't be more than a finite number of zeros for $f_+'(s)$. On $-a \leq s \leq 0$, $f_+'(s)$ has a root only if $U^2 < gh$ and in that case it is in $(-a/\sqrt{3}, 0)$.

Since $f_+(s) = -f_-(-s) = 0$, $f_-(s)$ has the same number of stationary points situated symmetrical to those of $f_+(s)$ on the negative real axis.

In order to consider the integral $\int_{-a}^{-r_1} (s+a)^{n-1/2} e^{isx} e^{-itf_+} ds$, we break the interval in such a way that the first subinterval, say, $(-a,\delta)$ does not have any stationary point of $f_+(s)$ in it, and each of the subsequent ones has only one such stationary point in its interior. Let one such interval be (α,β) such that $\alpha < \gamma < \beta$ and $f_+'(\gamma) = 0$.

If m is its multiplicity,

$$\int_{\alpha}^{\beta} (s+a)^{n-\frac{1}{2}} e^{isx} e^{-itf_+} ds$$

$$= (\gamma+a)^{n-\frac{1}{2}} e^{i\gamma x} \left(\frac{2\pi}{t\left|f_+^{m+1}(\gamma)\right|} \right)^{\frac{1}{m+1}} + O\left(t^{-\frac{m}{m+1}}\right)$$

$$\to \quad 0 \quad \text{as} \quad t \to \infty \ .$$

Similarly, the contribution from other stationary points to the first term of the asymptotic expansion of this integral vanishes. Finally to consider $\int_{-a}^{\delta} (s+a)^{n-\frac{1}{2}} e^{isx} e^{-itf_+} ds$. Let $s + a = \lambda^2$. Then

$$\int_0^{\sqrt{\delta+a}} \lambda^{2n-1}\, e^{ix\lambda^2}\, e^{-ixa} \exp\{-it[U(\lambda+a^2)+$$

$$+ \lambda\sqrt{(\lambda^2-a)(2a-\lambda^2)\tanh h(\lambda^2-a}\,\}\, 2\lambda\, d\lambda\ .$$

The major contribution to this nttegral comes from the stationary points of $U(a+\lambda^2)$ + $+ \lambda\sqrt{(\lambda^2-a)(2a-\lambda^2)\tanh h(\lambda^2-a)}$ i.e. from the values of λ satisfying $f_+'(\lambda^2-a) = 0$. As discussed before, this equation does not have any root because $\lambda^2 < a$ and $U^2 > gh$. Hence the first term in the esymptotic expansion of this integral equals $0(1/t) \to 0$ as $t \to \infty$. Similarly, the integrals in the second parentheses in equation (13) approach 0 as $t \to \infty$. Finally, we have:

$$\lim_{t\to\infty}\left[\int_{-a}^{-r_1} \frac{b_0}{\sqrt{s+a}}\, e^{isx}\, e^{-itf_+}\, ds\right.$$

$$\left. + \int_{-a}^{-r_1} \frac{c_0}{\sqrt{s+a}}\, e^{isx}\, e^{-itf_+}\, ds\right]\ .$$

If we carry out the substitution $\lambda^2 = s+a$, the resulting integrals are of the same type as discussed earlier. We can, therefore, conclude that they vanish for $t \to \infty$. Hence $I_{-a} \to 0$ as $t \to \infty$. Similarly, $I_a \to 0$ as $t \to \infty$.

$$I_\infty = \lim_{t\to\infty} \int_a^\infty \frac{is^2U\bar{\Pi}\cosh s(y+h)\, e^{isx}}{\rho \cosh sh}$$

$$\left[\frac{1}{s^2U^2+s(\frac{T}{\rho}s^2-g)\tanh sh} - \frac{1}{2\sqrt{\frac{T}{\rho}s(s^2-a^2)\tanh sh}} \right.$$

$$\left. \left\{ \frac{e^{-tf_+^{(1)}}}{f_+^{(1)}} - \frac{e^{-tf_-^{(1)}}}{f_-^{(1)}} \right\} \right] ds \ ,$$

where

$$f_\pm^{(1)} = isU \pm \sqrt{s(s^2\frac{T}{\rho}-g)\tanh sh} \ .$$

Let

$$(18) \qquad I'_\infty = \int_a^\infty \frac{is^2U\bar{\Pi}\cosh s(y+h)e^{isx}}{2\rho\cosh sh\sqrt{s\frac{T}{\rho}(s^2-a^2)\tanh sh}} \frac{e^{-tf_-^{(1)}}}{f_-^{(1)}} ds.$$

For the convergence of the integral, we observe that the singularity of the integrand at the lower end point is an integrable singularity and so is inconsequential. For convergence at the upper end point has been discussed before i.e. it is made possible by the presence of the factor e^{-s^2} in $\bar{\Pi}(s)$.

Let $s = a + t^3z^2$. Then

$$I_{\infty}^{1} =$$

$$\int_0^{\infty} \left[\frac{iU(a+t^3z^2)^2 \sqrt{(a+t^3z^2)}\cosh\{(a+t^3z^2)(y+h)\}e^{ix(a+t^3z^2)}}{2\rho\cosh(a+t^3z^2)h\{\frac{T}{\rho}(a+t^3z^2)t^3z^2(2a+t^3z^2)\tanh h(a+t^3z^2)\}^{1/2}} \right.$$

$$\left. \left[\frac{e^{-t\{iU(a+t^3z^2 - \sqrt{(a+t^2z^2)\frac{T}{\rho}t^3z^2(2a+t^3z^2)\tanh h(a+t^3z^2)}\}}}{iU(a+t^3z^2) - \sqrt{(a+t^3z^2)\frac{T}{\rho}t^3z^2(2a+t^3z^2)\tanh h(a+t^3z^2)}} \right] 2t^3 z\, dz. \right.$$

$$= i\, e^{-iUta} \frac{3}{t^2} \int_0^{\infty} \phi(z,t) \cdot e^{-iUt^4z^2 + ix(a+t^3z^2)}\, dz$$

where

$$\phi(z,t) = \left[\left\{ \frac{U(a+t^3z^2)^2\, g(a+t^3z^2)\cosh\{(a+t^3z^2)(y+h)\}}{\rho\cosh(a+t^3z^2)h\{\frac{T}{\rho}(a+t^3z^2)(2a+t^3z^2)\tanh h(a+t^3z^2)\}} \right\}^{1/2} \right.$$

$$\left\{ \frac{1}{iU(a+t^3z^2) - \sqrt{\frac{T}{\rho}t^3z^2(a+t^3z^2)(2a+t^3z^2)\tanh h(a+t^3z^2)}} \right\}$$

$$\left. \left\{ e^{-(a+t^3z^2)^2 t^{11/2} z \sqrt{\frac{T}{\rho}(z^2+\frac{a}{t^3})(z^2+\frac{2a}{t^3})\tanh h(a+t^3z^2)}} \right\} \right]$$

$$= \phi_1 \cdot \phi_2 \cdot \phi_3 \cdot$$

The function $\phi(z,t)$ is bounded for all positive t and is differentiable in the interval $[0,\infty)$.

$$\phi_2 = \frac{1}{iU(a+t^3z^2) - \sqrt{\frac{T}{\rho}\, t^3z^2(a+t^3z^2)(2a+t^3z^2)\tanh h(a+t^3z^2)}}$$

$$= \frac{-iU(a+t^3z^2) - \sqrt{\frac{T}{\rho}\, t^3z^2(a+t^3z^2)(2a+t^3z^2)\tanh h(a+t^3z^2)}}{U^2(a+t^3z^2)^2 + \frac{T}{\rho}\, t^3z^2(a+t^3z^2)(2a+t^3z^2)\tanh h(a+t^3z^2)}$$

$$= -i\phi_2' - \phi_2'' \cdot it^{3/2} e^{-iUta} \cdot$$

$$\int \phi(z,t)\, e^{i(-Ut^4z^2+xa+xt^3z^2)}\, dz = t^{3/2} e^{-iUta} \cdot$$

$$\int \phi'\, e^{i(-Ut^4z^2+xa+xt^3z^2)}\, dz - it^{3/2} e^{-iUta} \int \phi''\, e^{i(-Ut^4z^2+xa+xt^3z^2)}\, dz$$

where

$$\phi' = \phi_1 \cdot \phi_2' \cdot \phi_3 , \qquad \phi'' = \phi_1 \cdot \phi_2'' \cdot \phi_3 .$$

We have, therefore, to consider the two integrals

$$e^{-iUta} t^{3/2} \int_0^{\infty} \begin{Bmatrix} \phi' \\ \phi'' \end{Bmatrix} e^{i(-Ut^4z^2+xa+xt^3z^2)}\, dz$$

$$= e^{-iUta} t^{3/2} \int_0^{\sqrt{T/\rho}} \begin{Bmatrix} \phi' \\ \phi'' \end{Bmatrix} e^{i(-Ut^4z^2+xa+xt^3z^2)}\, dz \quad ;$$

$$+e^{-iUta}\, t^{3/2}\int_{\sqrt{T/\rho}}^{\infty}\begin{Bmatrix}\phi'\\ \phi''\end{Bmatrix} e^{i(-Ut^4z^2+xa+xt^3z^2)}\,dz$$

$$= J_1 + J_2$$

where

$$J_1 = e^{-iUta}\, t^{3/2}\int_0^{\sqrt{T/\rho}}\begin{Bmatrix}\phi'\\ \phi''\end{Bmatrix} e^{i(-Ut^4z^2+ax+xt^3z^2)}\,dz\ .$$

The sign of ϕ', ϕ'' depends upon the sign of the function g. We write $J_1' = e^{-iUta}\, t^{3/2}\int_0^{\sqrt{T/\rho}} \phi'\ e^{i(-Ut^4z^2+ax+xt^3z^2)}\,dz$.

The function ϕ' being analytic will have isolated and, therefore finite in number, zeros in the interval $[0, \sqrt{T/\rho}]$. Also the number of points in $[0, \sqrt{T/\rho}\ ]$ at which ϕ' attains maxima or minima will be finite. These two observations together imply that it is possible to divide $[0, \sqrt{T/\rho}\ ]$ into a finite number of such intervals such that ϕ' is monotonic in each of them and maintains the sign. Let one such interval be (α,β). The contribution to J_1' from this interval is:

$$I = e^{-iUta}\, e^{iax}\, t^{3/2}\int_\alpha^\beta \phi'\ e^{i(-Ut^4+xt^3)z^2}\,dz.$$

There are four cases:

(1) ϕ' is positive and monotonically decreasing in (α,β)

(2) ϕ' is negative and monotonically increasing in (α,β)

(3) ϕ' is positive and monotonically increasing

(4) ϕ' is negative and monotonically decreasing.

Case 1

$$I = e^{-iUta}\, e^{iax}\, t^{3/2} \int_{\alpha}^{\beta} \phi'\, e^{i(-Ut^4+xt^3)z^2}\, dz$$

$$+ e^{iax-iUta}\, t^{3/2} \phi'(\alpha) \int_{\alpha}^{\xi} e^{i(-Ut^4+xt^3)z^2}\, dz\ ,$$

$$\text{for } \alpha < \xi < \beta$$

and

$$\int_{\alpha}^{\xi} e^{i(-Ut^4+xt^3)z^2}\, dz = \begin{cases} \dfrac{e^{i(-Ut^4+xt^3)\xi^2}}{2\xi i(-Ut^4+xt^3)} - \dfrac{e^{i(-Ut^4+xt^3)\alpha^2}}{2i\alpha(-Ut^4+xt^3)} \\ \qquad + O\left(\dfrac{1}{(-Ut^4+xt^3)^2}\right) \qquad \text{for } \alpha \neq 0 \\ \left\{\dfrac{\pi}{4(Ut^4-xt^3)}\right\}^{1/2} e^{-\frac{1}{4}\pi i} \\ \qquad + O\left(\dfrac{1}{(Ut^4-xt^3)}\right) \qquad \text{for } \alpha = 0. \end{cases}$$

Also $\lim_{t\to\infty} \phi'(\alpha,t) = \text{finite}$. Hence $I \to 0$ as $t \to \infty$.

In case (2), we consider $-I = e^{i(ax-Uta)}\, t^{3/2} \int_{\alpha}^{\beta} (-\phi')\, e^{i(-Ut^4+xt^3)z^2}\, dz$. Since $-\phi'$ is positive and monotonically decreasing, it follows from case (1),

that $-I \to 0$ as $t \to \infty$.

In case (3), we have,

$$-I = t^{3/2} e^{iax-iUta} \int_{\alpha}^{\beta} \left\{ \phi'(\beta) - \phi'(z) \right\} e^{i(-Ut^4+xt^3)z^2} \, dz$$

$$-t^{3/2} e^{-i(ax-Uta)} \int_{\alpha}^{\beta} \phi'(\beta) \, e^{i(-Ut^4+xt^3)z^2} \, dz \, .$$

In the limit $t \to \infty$, the first integral on the right vanishes as in case (1).

From the asymptotic approximation of the second integral, it is clear that the second integral also tends to zero as $t \to \infty$. Case (4) reduces to case (3) if we consider it as

$$-I = e^{iax-iUta} t^{3/2} \int_{\alpha}^{\beta} (-\phi') e^{i(-Ut^4+xt^3)z^2} \, dz$$

and therefore can be treated likewise.

As (α,β) was any arbitrary subinterval, the same can be concluded for all such intervals. Hence $J_1' \to 0$ as $t \to \infty$. Similarly $J_1'' \to 0$ as $t \to \infty$.

We now consider

$$J_2 = e^{-iUta} t^{3/2} \int_{\sqrt{T/\rho}}^{\infty} \left\{ \begin{matrix} \phi' \\ \phi'' \end{matrix} \right\} e^{i(-Ut^4z^2+ax+xt^3z^2)} \, dz$$

$$= it^{3/2} e^{iax-iUta} \int_{\sqrt{T/\rho}}^{\infty} \phi(z,t) \, e^{i(-Ut^4+xt^3)z^2} \, dz \, .$$

$$|J_2| \le t^{3/2} \int_{\sqrt{T/\rho}}^{\infty} |\phi(z,t)|\, dz$$

$$|\phi| \le |\phi_1| \cdot |\phi_3|$$

$$\phi_1 = Ut^6\left(\frac{a}{t^3} + z^2\right)^2 \quad \frac{g \cosh\{(a+t^3z^2)(y+h)\}}{\rho \cosh (a+t^3z^2)h} \cdot$$

$$\cdot \frac{1}{t^3\{\frac{T}{\rho}(\frac{a}{t^3} + z^2)(\frac{2a}{t^3} + z^2)\tanh h(a+t^3z^2)\}^{1/2}}$$

For large t,

$$|\phi_1| \approx Ut^3z^2|g|e^{(a+t^3z^2)y} \frac{1}{\rho}\sqrt{\frac{\rho}{T}} \le t^3z^2 \frac{M}{\rho}\sqrt{\frac{\rho}{T}}$$

where $|g| \le M/U$ for all z in $[0,\infty)$.

$$|\phi_3| \approx e^{-t^6z^4} e^{\sqrt{T/\rho}\, z^3 t^{11/2}}$$

$$|J_2| \le \frac{t^{3/2}}{\rho} \int_{\sqrt{T/\rho}}^{\infty} t^3z^2 M \sqrt{\frac{\rho}{T}}\, e^{-t^6z^4 + \sqrt{T/\rho}\, z^3 t^{11/2}} dz$$

$$\le \frac{t^{9/2}}{\rho} M \int_{\sqrt{T/\rho}}^{\infty} z^2 \sqrt{\frac{\rho}{T}}\, e^{-t^6z^4 + \sqrt{T/\rho}\, t^6z^3} dz .$$

Let $z = \sqrt{T/\rho}\, \ell$, $dz = \sqrt{T/\rho}\, \ell$.

$$|J_2| \le \frac{t^{9/2}M}{\rho} \int_1^\infty \left(\frac{T}{\rho}\right) \ell^2 \, e^{t^6(T/\rho)^2(\ell^3-\ell^4)} \, d\ell$$

$$= \frac{t^{9/2}M}{\rho} \left(\frac{T}{\rho}\right) \lim_{L\to\infty} \int_1^L \ell^2 \, e^{t^6(T/\rho)^2(\ell^3-\ell^4)} \, d\ell$$

$$= \frac{M}{T\, t^{3/2}} \left[\lim_{L\to\infty} \int_1^L \left\{ \left(t^6\left(\frac{T}{\rho}\right)^2(3\ell^2-4\ell^3)\right) e^{t^6(T/\rho)^2(\ell^3-\ell^4)} \right\} \cdot \frac{1}{3-4\ell} \, d\ell \right]$$

$$= \frac{M}{Tt^{3/2}} \left| \lim_{L\to\infty} \left\{ \frac{e^{t^6(T/\rho)^2(\ell^3-\ell^4)}}{3-4\ell} \right|_1^L - 4 \int_1^L \frac{e^{t^6(T/\rho)^2(\ell^3-\ell^4)}}{(3-4\ell)^2} \, d\ell \right\} \right]$$

$$= \frac{M}{Tt^{3/2}} \lim_{L\to\infty} \left\{ 1 - \mu\left(\frac{1}{3-4L} + 1\right) \right\}, \quad \text{where } 0 < \mu < 1$$

$$= \frac{1-\mu}{T\, t^{3/2}} \to 0 \quad \text{as } t \to \infty$$

i.e. $I'_\infty \to 0$ as $t \to \infty$. We then consider

$$I''_\infty = -\int_a^\infty \frac{i\, s^2 U \cosh s(y+h)\, e^{isx}}{2\rho \cosh sh \sqrt{s\frac{T}{\rho}(s^2-a^2)\tanh sh}} \cdot \frac{e^{-tf_+^{(1)}}}{f_+^{(1)}} \, .$$

$$|I''_{\infty}| \leq \int_a^{\infty} \frac{s^2 U |\overline{\overline{\Pi}}| \cosh s(y+h)}{2\sqrt{\rho T}\cosh sh \sqrt{s(s^2-a^2)\tanh sh}}$$

$$\cdot \frac{e^{-t\sqrt{s(s^2\frac{T}{\rho}-g)\tanh sh}}}{|f_+^{(1)}|} ds .$$

This integral is uniformally convergent for all $t > 0$. Hence

$$\lim_{t\to\infty} \int_a^{\infty} \frac{s^2 U |\overline{\overline{\Pi}}| \cosh s(y+h)}{2\sqrt{\rho T}\cosh sh \sqrt{s(s^2-a^2)\tanh sh}}$$

$$\cdot \frac{e^{-t\sqrt{s(s^2\frac{T}{\rho}-g)\tanh sh}}}{|f_+^{(1)}|} ds$$

$$= \int_a^{\infty} \lim_{t\to\infty} \left[\frac{s^2 U |\overline{\overline{\Pi}}| \cosh s(y+h)\, e^{-t\sqrt{s(s^2\frac{T}{\rho}-g)\tanh sh}}}{2\sqrt{\rho T}\cosh sh \sqrt{s(s^2-a^2)\tanh sh}\ |f_-^{(1)}|} \right] ds$$

$$= 0 .$$

Hence $I''_{\infty} \to 0$ as $t \to \infty$. Similarly we can prove that the time-dependent part in $I_{-\infty} \to 0$ as $t \to \infty$.

It follows from (11), therefore, that time-dependent terms approach zero in the limit as $t \to \infty$, i.e. in case $U^2 > gh$, the steady flow exists. In order to find the relevant radiation conditions, we collect the remaining terms. We have then,

$$\phi_x = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{is^2 U \bar{\Pi} e^{isx} \cosh s(y+h)\, ds}{\rho \cosh sh(s^2U^2 - \sqrt{\frac{T}{\rho}}\, s(a^2-s^2)\tanh sh)}.$$

We have to find $\lim_{\substack{x\to +\infty \\ -\infty}} \phi_x$. Consider

$$(19) \int_{-\infty}^{\infty} \left[\frac{iU \cosh s(y+h)}{\cosh sh}\, ds\right] = 2\int_{0}^{\infty} \frac{iU \cosh s(y+h)}{\cosh sh}\, ds$$

$$\frac{\cosh s(y+h)}{\cosh sh} = \frac{e^{s(y+h)} + e^{-s(y+h)}}{e^{sh} + e^{-sh}} = e^{sy} \frac{1+e^{-2s(y+h)}}{1 + e^{-sh}}.$$

We observe that $s \geq 0$ and $y + h > 0$

$$e^{sy} \frac{1 + e^{-2s(y+h)}}{1 + e^{-2sh}} < 2e^{sy}$$

since y is negative, (19) converges absolutely.

The remaining function is

$$(20) \quad \frac{\bar{\Pi}(s)s^2}{s^2U^2 - s(g - s^2 \frac{T}{\rho}) \tanh hs} \quad \text{in} \quad (-\infty, \infty)$$

The singularity of the function is a double singularity at $s = 0$ which is neutralized by the factor s^2 in the

numerator. This implies that the function (19) is bounded. Hence

$$\int_{-\infty}^{\infty} \frac{is^2U\overline{\overline{\prod}}\cosh s(y+h)}{\rho \cosh sh(s^2U^2 - s(g-s^2\frac{T}{\rho}) \tanh sh)}\, ds$$

is absolutely convergent. By the Riemann-Lebesgue lemma, therefore,

$$\lim_{|x|\to\infty} \int_{-\infty}^{\infty} \frac{is^2U\overline{\overline{\prod}}\cosh s(y+h)\, e^{isx}}{\rho \cosh sh(s^2U^2 - s(g-s^2\frac{T}{\rho}) \tanh sh)}\, ds = 0 .$$

Also if we had worked with ϕ_y instead of working with ϕ_x, the only difference is that we would have sinh s(y+h) instead of cosh s(y+h). Exactly the same analysis holds in this case, which implies that $\lim_{|x|\to\infty} \phi_y = 0$, i.e. the disturbance dies out at $\pm\infty$. Hence the conclusion is that in case $U^2 > gh$, the steady solution does exist and the corresponding radiation conditions are that the disturbance dies out both upstream and downstream.

Case II $U^2 = gh$

Again

$$\phi(x,y;t) = \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} A(s,t) \cosh s(y+h)\, e^{isx}\, ds \right.$$

$$\left. + \int_{-\infty}^{-a} A(s,t) \cosh s(y+h)\, e^{isx}\, ds + \int_{a}^{\infty} A(s,t) \cosh s(y+h) s^{isx} ds \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} \frac{isU\overline{\mp}\cosh s(y+h) e^{isx}}{\rho \cosh sh} \left[\frac{1}{s^2U^2 - s(g - s^2 \frac{T}{\rho}) \tanh sh} \right.\right.$$

$$\left.\left. + \frac{1}{2\sqrt{(g - s^2 \frac{T}{\rho}) s \tanh sh}} \left\{ \frac{e^{-itf_+}}{f_+} - \frac{e^{-itf_-}}{f_-} \right\} \right] ds \right.$$

$$+ \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{-a} + \int_{a}^{\infty} \right) \left[\frac{isU\overline{\mp}\cosh s(y+h) e^{isx}}{\rho \cosh sh} \left\{ \frac{1}{s^2U^2 + s(-g + s^2\frac{T}{\rho}) \tanh sh} \right.\right.$$

$$\left.\left. + \frac{1}{2\sqrt{s(s^2 \frac{T}{\rho} - g) \tanh sh}} \left(\frac{e^{-tf_-^{(1)}}}{f_-^{(1)}} - \frac{e^{-tf_+^{(1)}}}{f_+^{(1)}} \right) \right\} \right] ds \, .$$

$$f_{\pm} = sU \pm \sqrt{s(g - s^2 \frac{T}{\rho}) \tanh sh}$$

$$f_{\pm}^{(1)} = isU \pm \sqrt{s(-g + s^2 \frac{T}{\rho}) \tanh sh}$$

$$= (\phi_a + \phi_{at}) + (\phi_\infty + \phi_\infty t) + (\phi_{-\infty} + \phi_{-\infty} t) \, .$$

We write

$$\phi_{at}^{t} = + \frac{U}{\sqrt{2\pi}} \int_{-a}^{a} \frac{\overline{\prod} s \cosh s(y+h)\, e^{isx}}{\sqrt{\rho T}\ \sqrt{s(a^2-s^2)\tanh sh}} (e^{-itf_+} - e^{-itf_-}) ds.$$

The main contribution to the asymptotic expansion of the integral ϕ_{at}^{t} comes from the stationary points of $f_{\pm}(s)$ and from the endpoints. We have already proved in a similar integral that the contribution from the endpoints tends to zero as $t \to \infty$. For the contribution from the stationary points, we study

$$\phi_{at}^{t} = \frac{U}{\sqrt{2\pi}} \int_{-a}^{a} \frac{\overline{\prod} s \cosh s\,(y+h)\, e^{isx}}{\sqrt{\rho T}\ \sqrt{s(a^2-s^2)\ \tanh hs}} \left(e^{-itf_+} - e^{-itf_-}\right) ds.$$

$f_-'(s)$ has a double zero at the origin and $f_+'(s)$ does not have any zero at the origin. Let us suppose that all other zeros of $f_{\pm}'(s)$ are simple. Then

$$\phi_{at}^{t} = \frac{+\ 2U}{\sqrt{2\pi\rho T}} \left[\frac{\overline{\prod}(0)}{a\ \sqrt{h}}\right] \left(\frac{2\pi}{t|f'''(0)|}\right)^{1/3} + O\left(\frac{1}{t^{2/3}}\right).$$

i.e.

$$\phi_{at}^{t} \quad O(t^{-1/3})$$

$$\phi_{at} \quad O(t^{2/3}) \quad \text{which} \to \infty \quad \text{as } t \to \infty .$$

If $f_{\pm}'(s)$ has a zero of order higher than 2. the rate at which $\phi_{at} \to \infty$ will be accordingly higher.

On the other hand, the contribution from $I_{-\infty\, t}$, $I_{\infty\, t}$ has already been seen to tend to zero for $t \to \infty$. Hence in the critical case, there does not exist a steady motion. The velocity potential $\to \infty$ at least as $t^{2/3}$. For any discussion of this case, we have to revert to linear theory.

Case III: $U^2 < gh$

$$\phi_x = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(s,t)\, s \cosh s(y+h) e^{isx}\, ds$$

$$= \frac{i}{\sqrt{2\pi}} \left[\left\{ \int_{-\infty}^{-a} + \int_{-a}^{a} + \int_{a}^{\infty} \right\} A(s,t) s \cosh s(y+h) e^{isx}\, ds. \right]$$

$$= \frac{i}{\sqrt{2\pi}} \left(I_{-\infty} + I_a + I_{\infty} \right)$$

$$I_a = \int_{-a}^{a} A(s,t) s \cosh s(y+h) e^{isx}\, ds$$

$$= \int_{-a}^{a} \frac{i s^2 U \sqrt{\quad}}{\rho \cosh s h} \left[\frac{1}{s^2 U^2 - s(g - s^2 \frac{T}{\rho}) \tanh sh} + \right.$$

$$\left. + \frac{1}{2\sqrt{s(g-s^2(T/\rho))\tanh sh}} \left(\frac{e^{-if_+ t}}{f_+} - \frac{e^{-if_- t}}{f_-} \right) \right] \cdot$$

$$\cdot \cosh s(y+h) e^{isx}\, ds.$$

$$f_{\pm} = s U \pm \sqrt{s(a^2 - s^2) \tanh sh}\, \sqrt{\frac{T}{\rho}}\,.$$

In this case, we know that both $f_{\pm}$ has a simple zero each at $s = 0$. Besides $f_-(\beta) = 0 = f_+(-\beta)$.

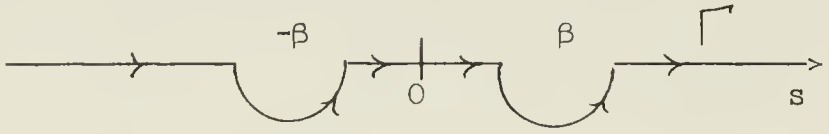
We write $I_a = I_{as} + I_{at}$,

where

$$I_{at} = \int_{-a}^{a} \frac{i s^2 U \sqrt{\quad}}{2\rho \cosh sh} \frac{e^{isx} \cosh s(y+h)}{\sqrt{s(g - s^2 \frac{T}{\rho}) \tanh sh}} \left(\frac{e^{-if_+ t}}{f_+} - \frac{e^{-if_- t}}{f_-} \right) ds.$$

Owing to the presence of the factor s^2 with numerator, $s = 0$ is not a singularity of the integrand.

In order to consider the singularities at $s = \pm \beta$, we deform the path of integration at $s = \beta, -\beta$ with two semicircles in the lower half plane as shown in the figure.



$$f_- = sU - \sqrt{(gs - s^3 \frac{T}{\rho}) \tanh hs}$$

$$f_-' = U - \frac{1}{2}\left[\frac{(g - 3s^2 \frac{T}{\rho}) \sqrt{\tanh sh}}{\sqrt{(gs - s^3)\frac{T}{\rho}}} + \frac{h \operatorname{sech}^2 sh}{\sqrt{\tanh sh}} \sqrt{gs - s^3 \frac{T}{\rho}} \right]$$

$$= U - \frac{1}{2}\left[\frac{(g - 3s^2 \frac{T}{\rho}) \tanh sh + h \operatorname{sech}^2 hs (gs - s^3 \frac{T}{\rho})}{\sqrt{(gs - s^3 \frac{T}{\rho}) \tanh sh}} \right] .$$

Also

$$\beta U^2 = (g - \beta^2 \frac{T}{\rho}) \tanh \beta h$$

$$f_-'(\beta) = U - \frac{1}{2}\left\{ \frac{\beta U^2 - 2\beta^2 \frac{T}{\rho} \tanh \beta h + h \operatorname{sech}^2 \beta h \, \beta^2 U^2 \coth \beta h}{\beta U} \right\}$$

$$= \frac{1}{2U}\left\{ U^2 + 2\beta \frac{T}{\rho} \tanh \beta h - \frac{2hU^2\beta}{\sinh 2h\beta} \right\}$$

$$= \frac{1}{2U}\left\{ U^2 \left(1 - \frac{2h\beta}{\sinh 2h\beta}\right) + 2\beta \frac{T}{\rho} \tanh h\beta \right\} > 0 .$$

Similarly

$$f_+(s) = sU + \sqrt{(gs - s^3\frac{T}{\rho})\tanh hs}$$

$$f_+'(-\beta) = U - \frac{1}{2}\,\frac{\{\beta U^2 - 2\beta^2\frac{T}{\rho}\tanh h\beta + h\,\mathrm{sech}^2 h\beta\,\beta^2U^2\coth\beta h\}}{\beta U}$$

which, before, is > 0.

Over the semicircles $s-\beta = r_1e^{i\theta}$, $s+\beta = r_1e^{i\phi}$, r, r_1 small and $\pi < \theta$, $\phi < 2\pi$, $f_-(s)$ and $f_+(s)$ behave like $f_-'(\beta)(s-\beta)$ and $f_+'(-\beta)(s+\beta)$ i.e. like $f_-'(\beta)re^{i\theta}$, $f_+'(-\beta)r_1e^{i\phi}$ respectively.

That is, the real parts of $\{-if_-(s)\}$, $\{-if_+(s)\}$ are negative.

We now have:

$$I_{at} = \int_\Gamma \frac{i\,s^2\,U\overline{\prod}}{\rho\,\cosh\,sh}\,\frac{e^{isx}\cosh\,s(y+h)}{2\sqrt{s(g-s^2\frac{T}{\rho})\,\tanh\,hs}}\,(\frac{e^{-itf_+}}{f_+} - \frac{e^{-itf_-}}{f_-})ds$$

$$= \int_\Gamma \frac{i\,s^2\,U\overline{\prod}}{\rho\,\cosh\,sh}\,\frac{e^{isx}\cosh\,s(y+h)}{2\sqrt{s(g-s^2\frac{T}{\rho})\tanh\,hs}}\,\frac{e^{-itf_+}}{f_+}\,ds$$

$$- \int_\Gamma \frac{i\,s^2\,U\overline{\prod}}{\rho\,\cosh\,sh}\,\frac{e^{isx}\cosh\,s(y+)}{2\sqrt{s(g-s^2\frac{T}{\rho})\tanh\,sh}}\,\frac{e^{-itf_-}}{f_-}\,ds$$

We consider the first integral over the semicircle $s+\beta = r_1e^{i\phi}$ and the second integral over the semicircle $s-\beta = re^{i\theta}$. The two integrands do not have any singularity over the semicircles. They are, in fact,

analytical function there at and so bounded. Hence

$$\left| \int_{s+\beta=r_1 e^{i\phi}} \frac{i\, s^2\, U \prod}{\rho \cosh sh} \frac{e^{isx} \cosh s(y+h)}{2\sqrt{s(g-s^2\frac{T}{\rho})\tanh sh}} \frac{e^{-itf_+}}{f_+}\, ds \right|$$

$$\leq \int_{s+\beta=r_1 e^{i\phi}} M\, C^{-Re(tf_+)}\, |ds|$$

$$\to 0 \text{ as } t \to \infty$$

where M is a bound of the coefficient of e^{-itf_+} in the integrand on $s+\beta = r_1 e^{i\phi}$, $\pi < \phi < 2\pi$.

Similarly the second integral also vanishes. On the straight parts of Γ and also for $I_{\pm\infty\, t}$ we can proceed as before in the first two cases, to prove that as $t \to \infty$ these integrals vanish.

Hence the steady motion in this case does exist.

To find the radiation conditions at $\pm\infty$, we consider the remaining integral,

$$I_s = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{i\, s^2\, U \prod}{\rho \cosh sh} \frac{\cosh s(y+h) e^{isx}}{s^2U^2 - s(g-s^2\frac{T}{\rho})\tanh sh}\, ds.$$

Let us suppose that $x < 0$. The zeros of the denominator are at $s = 0$ (double zero), and $s = \beta, -\beta$. Owing to the presence of s^2 in the numerator, the only singularities are at $s = \pm\beta$. We again consider the contour Γ from $-\infty$ to ∞. On this contour the coefficient of e^{isx} is again

analytic function and so bounded. On the two semicircles,

$$e^{isx} = e^{ixr_1(\cos\phi + i\sin\phi)} \qquad \pi < \phi < 2\pi$$

and

$$e^{isx} = e^{ixr(\cos\theta + i\sin\theta)} \qquad \pi < \theta < 2\pi$$

This => that for $x < 0$, the real part of the exponent on these circles is negative. We can then argue as before to conclude that for $x \to \infty$, I_s vanishes. On the straight parts, the integrand has no singularities and the integral

$$(21)\int \frac{i\, s^2\, U \overline{\overline{\Pi}}\ \cosh s(y+h)}{\rho \cosh sh}\left[\frac{1}{s^2U^2 - s(g - s^2\frac{T}{\rho})\tanh sh}\right] ds$$

is absolutely convergent.

Hence an application of Riemann Lebesgue lemma shows that the above integral I_s vanishes.

It follows, therefore, that the disturbance vanishes upstream.

Finally, we consider $x > 0$

$$(22)\quad I(s) = \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{is^2\, U \overline{\overline{\Pi}} \cosh s(y+h)e^{isx}}{\rho \cosh sh[s^2U^2 - s(g - s^2\frac{T}{\rho})\tanh sh]}\, ds\ .$$

The integrand now has poles at $s = \beta, -\beta$.

We deform the path of integration at $s = \beta, -\beta$ by drawing semicircles in the lower half plane On the straight parts of the contour, argument similar to that given above shows that $I(s) = 0$ for large x. the only contribution that we get is from the semicircular parts, say, B_+, B_-. To get that we first complete the two circles so that

$$e^{isx} = e^{isr_1(\cos\phi + i\sin\phi)} \quad \text{on } B_-$$

$$= e^{isr(\cos\theta + i\sin\theta)} \quad \text{on } B_+ .$$

On $0 < \theta, \phi < \pi$, the real part of the indices of the exponential is negative. Hence when $x \to \infty$, $I(s) \to 0$. This shows that the integral over the semicircle is the same thing as that over the whole circles.

On the other hand, since the integrand is analytic function except for two poles at the center we can use Cauchy's residue theorem whence it follows that:

$$I = -i\sqrt{2\pi} \lim_{s\to\beta} \left\{ \frac{is^2U(\bar{\Pi}(s)e^{is\beta} + \bar{\Pi}(-s)e^{-i\beta})(s-\beta)\cosh s(y+h)}{\cosh sh(s^2U^2 - s(g - s^2\frac{T}{\rho})\tanh sh} \right\}$$

$$= \sqrt{2\pi}\ \beta^2 U \left\{ \bar{\Pi}(\beta)e^{i\beta x} + \bar{\Pi}(-\beta)e^{-i\beta x} \right\} \frac{\cosh \beta(y+h)}{\cosh \beta h} H$$

where

$$H = \lim_{s\to\beta} \frac{s-\beta}{s^2U^2 - s(s - s^2\frac{T}{\rho})\tanh sh} .$$

β is given by the equation

$$\beta U^2 = (g-\beta^2 \frac{T}{\rho}) \tanh \beta h \ .$$

For large x, therefore,

$$(23)\ \phi_x \approx \sqrt{2\pi}\ \beta^2 U \{\overline{\Pi}(\beta) e^{i\beta x} + \overline{\Pi}(-\beta) e^{-i\beta x}\} \frac{\cosh \beta(y+h)}{\cosh \beta h} H,$$

which obviously does not tend to zero for $x \to \infty$.

The disturbance therefore does not die downstream.

We can evaluate H explicitly.

$$H = \lim_{s \to \beta} \frac{s-\beta}{s^2U^2 - s(g-s^2 \frac{T}{\rho}) \tanh sh}$$

$$= \frac{1}{2\beta U^2 - (g-3\beta^2 \frac{T}{\rho}) \tanh \beta h - (g\beta - \beta^3 \frac{T}{\rho}) h \operatorname{sech}^2 h\beta}$$

$$= \frac{1}{2(g-\beta^2 \frac{T}{\rho}) \tanh \beta h - (g-3\beta^2 \frac{T}{\rho}) \tanh \beta h - (g\beta - \beta^3 \frac{T}{\rho}) h \operatorname{sech}^2 h\beta}$$

$$= \frac{1}{(g+\beta^2 \frac{T}{\rho}) \tanh h\beta) - (g\beta - \beta^3 \frac{T}{\rho}) h \operatorname{sech}^2 h\beta}$$

$$|\beta| \leq \sqrt{\frac{\rho g}{T}}\ h.$$

Hence, when $T \to \infty$, $\beta \to 0$; H varies as $1/\beta$ and $\phi_x \to 0$.

The disturbance, therefore, though not dying downstream in general for any finite T, $\to 0$ when $T \to \infty$.

We can now summarize the results. First, steady state exists both for $U^2/gh \gtrless 1$. For the critical case i.e. $U^2 = gh$, there is no such solution. Indeed, the velocity potential grows with t at least as $t^{2/3}$ and becomes infinite as $t \to \infty$. In case of supercritical case, the radiation conditions are that the disturbance should die both upstream and downstream. In the subcritical case, the disturbance tends to zero upstream, but does not, in general, do so downstream. When it does not tend to zero, the velocity at infinity is given by (23). If we take $T = 0$, we get the same result as obtained by Stoker [9]. On the other hand, the dependence of ϕ_x on surface tension T is inversely proportional. When $T \to \infty$, the disturbance dies downstream as well.

Since gravity being a finite force is negligible in comparison with arbitrarily large T, the above result holds even for non-gravitating fluids.

Chapter II

The Solitary and Cnoidal Waves

Formulation of the Problem

We consider two dimensional flow of liquid in an infinite channel of finite depth d. The assumptions on the liquid and the flow are the same as those in Chapter I. The pressure on the free surface, however, is supposed to be $p_o(x)$.

Let $\tilde{u}$, $\tilde{v}$ be the horizontal and vertical components of velocity at a point (x,y) at a time t after the disturbance has been generated. The equations governing the flow are:

$$\tilde{u}_t + \tilde{u}\,\tilde{u}_x + \tilde{v}\,\tilde{u}_y = -\frac{1}{\rho}\,p_x$$

$$\tilde{v}_t + \tilde{u}\,\tilde{v}_x + \tilde{v}\,\tilde{v}_y = -\frac{1}{\rho}\,p_y - g$$

$$\tilde{v}_x = \tilde{u}_y$$

The equation of continuity gives

$$\tilde{u}_x + \tilde{v}_y = 0$$

The kinematical conditions are:

$$\tilde{\eta}_t + \tilde{u}\,\tilde{\eta}_x = \tilde{v} \qquad \text{at } y = \tilde{\eta}(x,t)$$

$$\tilde{v} = 0 \qquad \text{at } y = d$$

The discontinuity at the free surface is given by

$$p - p_o = \frac{- T\tilde{\eta}_{xx}}{(1+\tilde{\eta}_x^2)^{3/2}} \qquad \text{at } y = \tilde{\eta}(x,t)$$

where $p_o(x)$ is the pressure of the atmosphere at a point $(x,\tilde{\eta})$ just outside the liquid and $p(x,\tilde{\eta})$ at the corresponding point inside the liquid. $\tilde{\eta}(x,t)$ is free surface elevation. We suppose that $p_o(x) \longrightarrow 0$ as $x \longrightarrow -\infty$. The problem then is to obtain the equation governing this unsteady wave motion by using the stretching and expansion procedure of Keller [7] and Friedrichs [2]. This equation is then to be solved for particular pressure distributions on the free surface.

We use the following transformations to change the independent and the dependent variables in the above equations.

$$x = \frac{\alpha}{\omega} \qquad y = \beta h \qquad \tilde{u} = \sqrt{gh}\, u \qquad \tilde{v} = \sqrt{gh}\, \frac{v}{h\omega}$$

$$\tau = \sqrt{gh}\, \omega^3 h^2 t \qquad \eta = hY \qquad d = hH \qquad \sigma = \omega^2 h^2$$

$$p = g\rho h\, \pi \qquad p_o = g\rho h\, \sigma^2 \pi_S$$

where h has the dimension of length and ω has the dimension of inverse length. The new quantities α, β, u, v, π, π_S, y, τ are dimensionless. We have then:

$$\tilde{u}_x = \sqrt{gh}\ \omega\ u_\alpha \qquad \tilde{u}_y = \sqrt{gh}\ u_\beta \cdot \frac{1}{h}$$

$$\tilde{u}_t = \sqrt{gh} \cdot \sqrt{gh}\ \omega^3\ h^2\ u_\tau = g\omega^3\ h^3\ u_\tau \qquad \tilde{v}_x = v_\alpha \sqrt{\frac{g}{h}}$$

$$\tilde{v}_y = \frac{\sqrt{gh}}{h^2\omega} \qquad \tilde{v}_t = g\omega^2\ h^2\ v_\tau \qquad p_x = \rho g h \omega \pi_\alpha$$

$$p_y = \rho g \pi_\beta \qquad \tilde{\eta}_t = Y_\tau\ \omega^3\ h^3 \sqrt{gh} \qquad \tilde{\eta}_x = h\ Y_\alpha\ \omega$$

$$T = g\rho h^2\ T_o$$

substituting these in the above equations we get,

$$\sigma\ u_\alpha + v_\beta = 0$$

$$\sigma^2\ u_\tau + \sigma\ u\ u_\alpha + v\ u_\beta + \sigma\ \pi_\alpha = 0$$

$$\sigma 2\ v_\tau + \sigma\ u\ v_\alpha + v\ v_\beta + \sigma\ \pi_\beta + \sigma = 0$$

$$v_\alpha = u_\beta$$

$$\pi - \sigma^2\ \pi_S = \frac{-\sigma\ Y_{\alpha\alpha}\ T_o}{(1+\sigma Y_\alpha^2)^{3/2}} \qquad \text{at } \beta = Y$$

$$v = 0 \qquad \text{at } \beta = H$$

$$\sigma^2\ Y_\tau + \sigma u\ Y_\alpha = v \qquad \text{at } \beta = Y$$

We assume the following expansions:

$$u = u_o + \sigma u_1 + \sigma^2 u_2 + \sigma^3 u_3 + \ldots$$

$$v = v_o + \sigma v_1 + \sigma^2 v_2 + \sigma^3 v_3 + \ldots$$

$$\pi = \pi_o + \sigma \pi_1 + \sigma^2 \pi_2 + \sigma^3 \pi_3 + \ldots$$

$$Y = Y_o + \sigma Y_1 + \sigma^2 Y_2 + \sigma^3 Y_3 + \ldots$$

Substituting these for u, v, π, Y in the above equations we get,

$$\sigma[u_{o\alpha} + \sigma u_{1\alpha} + \sigma^2 u_{2\alpha} + \sigma^3 u_{3\alpha} + \ldots] + v_{o\beta} + \sigma v_{1\beta} + \sigma^2 v_{2\beta} + \sigma^3 v_{3\beta} + \ldots = 0$$

$$\sigma^2[u_{o\tau} + \sigma u_{1\tau} + \sigma^2 u_{2\tau} + \sigma^3 u_{3\tau}]$$
$$+ \sigma[u_o + \sigma u_1 + \sigma^2 u_2 + \ldots][u_{o\alpha} + \sigma u_{1\alpha} + \sigma^2 u_{2\alpha} + \sigma^3 u_{3\alpha} + \ldots]$$
$$+ [v_o + \sigma v_1 + \sigma^2 v_2 + \sigma^2 v_3 + \ldots][u_{o\beta} + \sigma u_{1\beta} + \sigma^2 u_{2\beta} + \sigma^3 u_{3\beta} + \ldots]$$
$$+ \sigma[\pi_{o\alpha} + \sigma \pi_{1\alpha} + \sigma^2 \pi_{2\alpha} + \ldots] = 0$$

$$\sigma^2[v_{o\tau} + \sigma v_{1\tau} + \sigma^2 v_{2\tau} + \sigma^3 v_{3\tau} + \ldots]$$
$$+ \sigma[u_o + \sigma u_1 + \sigma^2 u_2 + \sigma^3 u_3 + \ldots][v_{o\alpha} + \sigma v_{1\alpha} + \sigma^2 v_{2\alpha} + \sigma^3 v_{3\alpha} + \ldots]$$
$$+ [v_o + \sigma v_1 + \sigma^2 v_2 + \sigma^3 v_3 + \ldots][v_{o\beta} + \sigma v_{1\beta} + \sigma^2 v_{2\beta} + \sigma^3 v_{3\beta} + \ldots]$$
$$+ \sigma[\pi_{o\beta} + \sigma \pi_{1\beta} + \sigma^2 \pi_{2\beta} + \sigma^3 \pi_{3\beta} + \ldots] + \sigma = 0$$

$$v_{o\alpha}+\sigma v_{1\alpha}+\sigma^2 v_{2\alpha}+\sigma^2 v_{3\alpha}+ \ldots = u_{o\beta}+\sigma u_{1\beta}+\sigma^2 u_{2\beta}+\sigma^3 u_{3\beta} + \ldots$$

$$[\pi_o+\sigma(\pi_1+Y_1\pi_{o\beta})+\sigma^2(\pi_2+Y_1\pi_{1\beta}+Y_2\pi_{o\beta}+\tfrac{1}{2} Y_1^2\pi_{o\beta\beta})+ \ldots] - \sigma^2\pi_S$$

$$= -\sigma T_o[Y_{o\alpha\alpha}+\sigma(Y_{1\alpha\alpha} - \tfrac{3}{2} Y_{o\alpha}^2 Y_{o\alpha\alpha})$$

$$+\sigma^2(Y_{2\alpha\alpha} - \tfrac{3}{2} Y_{o\alpha}^2 Y_{1\alpha\alpha} + \tfrac{15}{8} Y_{o\alpha}^4 Y_{o\alpha\alpha} - 3Y_{o\alpha} Y_{o\alpha\alpha} Y_{1\alpha})$$

$$+\sigma^3(Y_{3\alpha\alpha} - \tfrac{3}{2} Y_{o\alpha}^2 Y_{2\alpha\alpha} + \tfrac{15}{8} Y_{o\alpha}^4 Y_{1\alpha\alpha} + \tfrac{15}{2} Y_{o\alpha\alpha} Y_{o\alpha}^3 Y_{1\alpha\alpha}$$

$$- \tfrac{35}{16} Y_{o\alpha\alpha} Y_{o\alpha}^6 - \tfrac{3}{2} Y_{1\alpha}^2 Y_{o\alpha\alpha} - 3Y_{o\alpha} Y_{2\alpha} Y_{o\alpha\alpha}) + \ldots]$$

at $\beta = Y_o$

$$v_o + \sigma v_1 + \sigma^2 v_2 + \sigma^3 v_3 + \ldots = 0 \qquad \text{for } \beta = H$$

$$\sigma^2(Y_{o\tau}+\sigma Y_{1\tau}+\sigma^2 Y_{2\tau} + \ldots) + \sigma(Y_{o\alpha}+\sigma Y_{1\alpha}+\sigma^2 Y_{2\alpha}+\sigma^3 Y_{3\alpha} + \ldots) \times$$

$$\{u_o+\sigma(u_1+Y_1u_{o\beta}) + \sigma^2(u_2+Y_1u_{1\beta}+Y_2u_{o\beta}+ \tfrac{1}{2} Y_1^2 u_{o\beta\beta}) +$$

$$+\sigma^3(u_3+Y_1u_{2\beta}+Y_2u_{1\beta} + \tfrac{1}{2} Y_1^2 u_{1\beta\beta}+Y_3u_{o\beta}+Y_1Y_2u_{o\beta\beta}$$

$$+ \frac{Y_1^3}{6} u_{o\beta\beta\beta}) + \ldots\}$$

$$= v_o+\sigma(v_1+Y_1v_{o\beta})+\sigma^2(v_2+Y_1Y_{1\beta}+Y_2v_{o\beta} + \tfrac{1}{2} Y_1^2 v_{o\beta\beta})$$

$$+\sigma^3(v_3+Y_1v_{2\beta}+Y_2v_{1\beta}+ \tfrac{1}{2} Y_1^2 v_{1\beta\beta}+Y_3v_{o\beta}+Y_1Y_2v_{o\beta}+ \frac{Y_1^3}{6} v_{o\beta\beta\beta})+\ldots$$

at $\beta = Y_o$

Equating zero order terms we get,

(1) $$v_{o\beta} = 0$$

(2) $$v_o \, u_{o\beta} = 0$$

(3) $$v_o \, v_{o\beta} = 0$$

(4) $$v_{o\alpha} = u_{o\beta}$$

(5) $$\pi_o = 0 \qquad \text{at } \beta = Y_o$$

(6) $$v_o = 0 \qquad \text{at } \beta = H$$

(7) $$v_o = 0 \qquad \text{at } \beta = Y_o$$

implies $$v_o(\alpha,\beta,\tau) = 0 \qquad u_o = u_o(\alpha,\tau)$$

I order terms

(8) $$u_{o\alpha} + v_{1\beta} = 0$$

(9) $$u_o \, u_{o\alpha} + v_o \, u_{1\beta} + \pi_{o\alpha} = 0$$

(10) $$u_o \, v_{o\alpha} + v_o \, v_{1\beta} + \pi_{o\beta} + 1 = 0$$

(11) $$v_{o\alpha} = u_{o\beta}$$

(12) $$\pi_1 + Y_1 \pi_{o\beta} + T_o Y_{o\alpha\alpha} = 0 \qquad \text{at } \beta = Y_o$$

(13) $$v_1 = 0 \qquad \text{at } \beta = H$$

(14) $$u_o Y_{o\alpha} = v_1 + Y_1 v_{o\beta} \qquad \text{at } \beta = Y_o$$

The first equation yields:

$$v_1 = - \beta u_{o\alpha} + f(\alpha,\tau)$$

$$0 = - H u_{o\alpha} + f(\alpha,\tau)$$

(15) $$v_1 = (H - \beta) u_{o\alpha}$$

The third equation gives:

$$\pi_o = - \beta + f_1(\alpha,\tau)$$

$$0 = - Y_o + f_1(\alpha,\tau)$$

(16) $$\pi_o = Y_o - \beta$$

Substituting for π_o in (9)

$$u_o u_{o\alpha} = 0$$

$$u_o = f_2(\tau)$$

We assume that the disturbance at $\alpha = - \infty$ is independent of time τ.

Then $u_o = \text{const}$

Let $u(-\infty,\tau) = c$, a constant

$$c = u(-\infty,\tau) = u_o + \sigma u_1 + \ldots \Big|_{(-\infty,\tau)}$$

$$= c_o + \sigma c_1 + \sigma^2 c_2 + \ldots \tag{17}$$

Thus $u_o = c_o$

The last equation now yields:

$$Y_o = \text{constant} \tag{18}$$

Hence zero order terms yield uniform flow.

II order terms:

$$u_{1\alpha} + v_{2\beta} = 0$$

$$u_{o\tau} + u_o\, u_{1\alpha} + u_1\, u_{o\alpha} + v_o\, u_{2\beta} + v_1\, u_{1\beta} + v_2\, u_{o\beta} + \pi_{1\alpha} = 0$$

$$v_{o\tau} + u_o\, v_{1\alpha} + u_1\, v_{o\alpha} + v_o\, v_{2\beta} + v_1\, v_{1\beta} + v_2\, v_{o\beta} + \pi_{1\beta} = 0$$

$$v_{2\alpha} = u_{2\beta}$$

We enjoin to these the following equations from the first order terms:

$$\pi_1 + Y_1 \pi_{o\beta} + T_o Y_{o\alpha\alpha} = 0 \qquad \text{at } \beta = Y_o$$

$$v_2 = 0 \qquad \text{at } \beta = H$$

$$Y_{o\tau} + u_o Y_{1\alpha} + Y_{o\alpha}(u_1 + Y_1 u_{o\beta}) = v_2 + Y_1 Y_{1\beta} + Y_2 v_{o\beta} + \frac{1}{2} Y_1^2 v_{o\beta\beta} \qquad \text{at } \beta = Y_o$$

Using the previous results, the above equations are reduced to

(19) $$u_{1\alpha} + v_{2\beta} = 0$$

(20) $$u_o u_{1\alpha} + \pi_{1\alpha} = 0$$

(21) $$\pi_{1\beta} = 0$$

(22) $$v_{2\alpha} = u_{2\beta}$$

(23) $$\pi_1 = Y_1 \qquad \text{at } \beta = Y_o$$

(24) $$v_2 = 0 \qquad \text{at } \beta = H$$

(25) $$u_o Y_{1\alpha} = v_2 \qquad \text{at } \beta = Y_o$$

From (21) and (23) we have $\pi_1 = Y_1$ everywhere. Integrating the equation (20) we get,

$$u_o\, u_1 + \pi_1 = f_3(\tau,\beta)$$

(26) i.e. $$u_o\, u_1 + Y_1 = f_3(\tau,\beta)$$

In view of the assumption that the disturbance at $\alpha = -\infty$ is independent of time, $f_3(\tau,\beta)$ in (26) must be a constant. In particular u_1 is independent of β.

$$v_{2\beta} = - u_{1\alpha}$$

$$v_2 = - \beta u_{1\alpha} + f_4(\alpha,\tau)$$

$$0 = - H u_{1\alpha} + f_4(\alpha,\tau)$$

(27) $$v_2 = (H - \beta)\, u_{1\alpha}$$

From (22) we then have:

$$u_{2\beta} = (H - \beta)\, u_{1\alpha\alpha}$$

$$u_2 = - \frac{1}{2} (H - \beta)^2\, u_{1\alpha\alpha} + F(\alpha,\tau)$$

From (25) and (27) we get,

$$u_o\, Y_{1\alpha} = (H - Y_o)\, u_{1\alpha}$$

Substituting for π_1 in (20)

$$u_o\, u_{1\alpha} + Y_{1\alpha} = 0$$

The two together imply that either u_1, Y_1 are constants or that

$$u_o^2 = Y_o - H \tag{28}$$

and then

$$Y_1 = - u_o u_1 + f_5(\tau)$$

Again evaluating this equation at $\alpha = - \infty$, we find $f_5(\tau)$ = a constant, say δ.

$$Y_1 = - u_o u_1 + \delta \tag{29}$$

Also since $u_o = c_o$, $v_1 = 0$

$$u_2 = - \frac{1}{2} (H - \beta)^2 u_{1\alpha\alpha} + F(\alpha,\tau)$$

III order terms:

$$u_{2\alpha} + v_{3\beta} = 0$$

$$u_{1\tau} + u_o u_{2\alpha} + u_1 u_{1\alpha} + u_2 u_{o\alpha} + v_o u_{3\beta} + v_1 u_{2\beta} + v_2 u_{1\beta} + v_3 u_{o\beta} + \pi_{2\alpha} = 0$$

$$v_{1\tau} + u_o v_{2\alpha} + u_1 v_{1\alpha} + u_2 v_{o\alpha} + v_o v_{3\beta} + v_1 v_{2\beta} + v_2 v_{1\beta} + v_3 v_{o\beta} + \pi_{2\beta} = 0$$

$$v_{3\alpha} = u_{3\beta}$$

$$v_3 = 0 \qquad \text{at } \beta = H$$

$$Y_{1\tau} + Y_{o\alpha}(\ ) + Y_{1\alpha}(u_1 + Y_1\, u_{o\beta}) + u_o\, Y_{2\alpha} = v_3 + v_{2\beta}\, Y_1$$

$$\text{at } \beta = Y_o$$

With these we take the pressure equation from the second order terms:

$$\pi_2 - \pi_S + Y_1\, \pi_{1\beta} + Y_2\, \pi_{o\beta} + \frac{1}{2}\, Y_1^2\, \pi_{o\beta\beta} = -\, T_o(Y_{1\alpha\alpha} - \frac{3}{2}\, Y_{o\alpha}^2\, Y_{o\alpha\alpha})$$

$$\text{at } \beta = Y_o$$

Making use of the previous results, the above equations can be reduced to

(30) $$u_{2\alpha} + v_{3\beta} = 0$$

(31) $$u_{1\tau} + u_o\, u_{2\alpha} + u_1\, u_{1\alpha} + \pi_{2\alpha} = 0$$

(32) $$u_o\, v_{2\alpha} + \pi_{2\beta} = 0$$

(33) $$v_{3\alpha} = u_{3\beta} \qquad \text{at } \beta = H$$

(34) $$v_3 = 0$$

(35) $$Y_{1\tau} + Y_{1\alpha}\, u_1 + u_o\, Y_{2\alpha} = v_3 - Y_1\, u_{1\alpha} \quad \text{at } \beta = Y_o$$

(36) $$\pi_2 - \pi_S - Y_2 + T_o\, Y_{1\alpha\alpha} = 0 \qquad \text{at } \beta = Y_o$$

$$\pi_{2\beta} + u_o \, v_{2\alpha} = 0$$

$$\pi_{2\beta} + u_o(H - \beta) \, u_{1\alpha\alpha} = 0$$

$$\pi_2 - \frac{u_o}{2} (\beta - H)^2 \, u_{1\alpha\alpha} = G(\alpha,\tau)$$

At $\beta = Y_o$ $\qquad \pi_2 = \pi_S + Y_2 - T_o \, Y_{1\alpha\alpha}$

$$= \frac{u_o}{2} (H - Y_o)^2 \, u_{1\alpha\alpha} + G(\alpha,\tau)$$

$$Y_2 = \frac{1}{2} u_o(H - Y_o)^2 \, u_{1\alpha\alpha} + G(\alpha,\tau) + T_o \, Y_{1\alpha\alpha} - \pi_S$$

$$u_2 = - \frac{1}{2} (H - \beta)^2 \, u_{1\alpha\alpha} + F(\alpha,\tau)$$

$$v_2 = (H - \beta) \, u_{1\alpha}$$

$$u_{1\tau} + u_o[F_\alpha - \tfrac{1}{2}(H - \beta)^2 \, u_{1\alpha\alpha\alpha}] + u_1 \, u_{1\alpha} + \pi_{2\alpha} = 0$$

Substituting for π_2 and $\beta = Y_o$ we get,

$$u_{1\tau} + u_o \, F_\alpha - \frac{1}{2} u_o(H - Y_o)^2 \, u_{1\alpha\alpha\alpha} + u_1 \, u_{1\alpha} + G_\alpha +$$

$$+ \frac{u_o}{2} (H - Y_o)^2 \, u_{1\alpha\alpha\alpha} = 0$$

i.e. $\qquad u_{1\tau} + u_1 \, u_{1\alpha} + G_\alpha + u_o F_\alpha = 0$

$$v_{3\beta} = - u_{2\alpha}$$

$$= - [F_\alpha - \frac{1}{2} (H - \beta)^2 \, u_{1\alpha\alpha\alpha}]$$

$$v_3 = - \beta F_\alpha - \frac{1}{6} (H - \beta)^3 u_{1\alpha\alpha\alpha} + \text{const}$$

$$0 = - H F_\alpha - \frac{1}{6} (H - H)^3 u_{1\alpha\alpha\alpha} + \text{const}$$

$$v_3 = (H - \beta) F_\alpha - \frac{1}{6} (H - \beta)^3 u_{1\alpha\alpha\alpha}$$

Eliminating v_3 from this equation and the equation (35)

$$u_o Y_{2\alpha} - (H - Y_o) F_\alpha = - u_1 Y_{1\alpha} - Y_{1\tau} - \frac{1}{6} (H - Y_o)^3 u_{1\alpha\alpha\alpha} - Y_1 u_{1\alpha}$$

$$u_o F_\alpha + G_\alpha = - u_{1\tau} - u_1 u_{1\alpha}$$

$$Y_{2\alpha} - G_\alpha = \frac{1}{2} u_o (H - Y_o)^2 u_{1\alpha\alpha\alpha} + T_o Y_{1\alpha\alpha\alpha} - \pi_{S\alpha} .$$

The determinant

$$\begin{vmatrix} u_o & Y_o - H & 0 \\ 0 & u_o & 1 \\ 1 & 0 & -1 \end{vmatrix} = 0$$

Hence these three equations considered as equations in F_α, G_α, $Y_{2\alpha}$ are not independent. Adding the last two equations we have:

$$Y_{2\alpha} + u_o F_\alpha = - u_{1\tau} - u_1 u_{1\alpha} + \frac{1}{2} u_o u_{1\alpha\alpha\alpha} (H - Y_o)^2 - \pi_{S\alpha} + T_o Y_{1\alpha\alpha\alpha}$$

Using the first equations we have:

$$u_o Y_{2\alpha} + u_o^2 F_\alpha = - u_o u_{1\tau} - u_o u_1 u_{1\alpha} + \frac{1}{2} u_o^2 u_{1\alpha\alpha\alpha}(H - Y_o)^2$$

$$- u_o \pi_{S\alpha} + u_o T_o Y_{1\alpha\alpha\alpha}$$

$$= - u_1 Y_{1\alpha} - Y_{1\tau} - \frac{1}{6}(Y_o - H)^3 u_{1\alpha\alpha\alpha} - Y_1 u_{1\alpha} .$$

(37) i.e. $$2Y_{1\tau} = \frac{\partial}{\partial\alpha}\left[Y_{1\alpha\alpha} u_o(\tfrac{1}{3} u_o^4 - T_o) + \frac{3}{2u_o} Y_1^2 - \frac{2\delta}{u_o} Y_1 + u_o \pi_S\right]$$

Which is the required differential equation, determining the propogation of disturbance. If we take the surface pressure p_o and therefore, π_S, constant, the resulting equation is similar to Korteweg- DeVries equation. The constant δ can be explicitly evaluated from the conditions at infinity. Thus if the disturbance is supposed to die at $\alpha = - \infty$, then from equations (29), (17) $\delta = c_o c_1$. We now study time-independent motion with two special pressure distributions:

(i) $$\pi_S = \begin{cases} 0 & \text{for } - \infty < \alpha < 0 \\ \text{constant} & \text{for } 0 \leq \alpha < \infty \end{cases}$$

(ii) $$\pi_S = \begin{cases} \text{constant} & \text{for } |\alpha| \leq a \\ 0 & \text{for } |\alpha| > a. \end{cases}$$

Also now $Y_{1\tau} = 0$.

Integrating (37) and using the fact that Y_1, $Y_{1\alpha\alpha}$, $\pi_S \longrightarrow 0$ as $\alpha \longrightarrow - \infty$

(38) $$Y_{1\alpha\alpha}\, K + \frac{3}{2c_o^2}\, Y_1^2 - \frac{2c_1}{c_o}\, Y_1 = -\,\pi_S$$

where $$K = \frac{1}{3}\, c_o^4 - T_o \ .$$

(i) The integration of (38) depends upon the sign of K.

(a) let $K > 0$, i.e. $\frac{1}{3}\, c_o^4 > T_o$

$$\frac{2}{3}\,(Y_o - H)^2 > T_o$$

i.e. $$(\eta_o - d)^2 > \frac{3T}{g\rho}$$

i.e. depth of the water in equilibrium should be greater than $\sqrt{\frac{3T}{g\rho}}$. In order to integrate (38) we first make the substitution

$$Y_1 = G - \left\{ \frac{-2}{3}\, c_1\, c_o + \frac{c_o}{3}\sqrt{4\,c_1^2 - 6\pi_S} \right\}$$

where $c_1^2 \geq \frac{3}{2}\,\pi_S$. If $c_1^2 < \frac{3}{2}\,\pi_S$, there is no equilibrium position about which disturbance can take place.

The equation becomes

$$K\, G_{\alpha\alpha} + \frac{3}{2c_o^2}\, G^2 - \frac{G}{c_o}\sqrt{4\,c_1^2 - 6\pi_S} = 0$$

The first integral is given by:

(39) $$K\, G_\alpha^2 + \frac{1}{c_o^2}\, G^3 - \frac{G^2}{c_o}\sqrt{4c_1^2 - 6\pi_S} = A$$

where A is the constant of integration.

Let $b = \sqrt{4c_1^2 - 6\pi_s}$ and $g = G - r$, r a constant.

The equation (39) becomes

$$K g_\alpha^2 + \frac{1}{c_o^2}(g^3 + 3g^2 r + 3gr^2 + r^3)$$

$$- \frac{1}{c_o} b(g^2 + 2rg + r^2) - A = 0$$

We choose r such that $\frac{r^3}{c_o^2} - \frac{br^2}{c_o} - A = 0$

i.e. $$r^3 - bc_o r^2 - Ac_o^2 = 0$$

$$K g_\alpha^2 + \frac{1}{c_o^2} g^3 + g^2\left(\frac{3r}{c_o^2} - \frac{b}{c_o}\right) + g\left(\frac{3r^2}{c_o^2} - \frac{2rb}{c_o}\right) = 0$$

Let $\frac{\lambda}{3} = -3r + bc_o \qquad \frac{\mu}{3} = 3r^2 - 2rbc_o$

$$K g_\alpha^2 + \frac{1}{c_o^2} g^3 - \frac{g^2\lambda}{3c_o^2} + \frac{g}{3c_o^2}\mu = 0$$

$$g_\alpha^2 = \frac{1}{c_o^2 K}\left(-g^3 + g^2\frac{\lambda}{3} - g\frac{\mu}{3}\right)$$

$$\pm \int \frac{c_o\sqrt{3K}\, dg}{\sqrt{-3g^3 + \lambda g^2 - g\mu}} = \alpha + B$$

B is a constant of integration.

$$\alpha + B = \pm \int \frac{c_o \sqrt{3K}\, dg}{\sqrt{-3g^3 + \lambda g^2 - g\mu}}$$

For $g = \frac{1}{T^2}$,

$$\alpha + B = \pm 2 \int \frac{c_o \sqrt{3K}\, dT}{\{-3 + \lambda T^2 - \mu T^4\}^{1/2}}$$

Let $\mu < 0$

$$\alpha + B = \pm 2c_o \sqrt{3K} \int \frac{dT}{[(-\mu)\{T^2 - (\frac{+\lambda}{2\mu} - \frac{1}{2\mu}\sqrt{\lambda^2 - 12\mu})\} \times \{T^2 + (\frac{\lambda}{-2\mu} + \frac{1}{-2\mu}\sqrt{\lambda^2 - 12\mu})\}]^{1/2}}$$

$$= \pm 2c_o \sqrt{3K} \left[\frac{1}{(\lambda^2 - 12\mu)^{1/4}} \, cn^{-1} \left\{ \frac{(\frac{+\lambda}{2\mu} - \frac{1}{2\mu}\sqrt{\lambda^2 - 12\mu})^{1/2}}{T} \right\} . \right.$$

$$\left. \text{mod} \left(\frac{\sqrt{+\lambda + \sqrt{\lambda^2 - 12\mu}}}{\sqrt{2}(\lambda^2 - 12\mu)^{1/4}} \right) \right]$$

$$(40) \qquad cn^2 \left\{ \frac{(\lambda^2 - 12\mu)^{1/4}}{2c_o \sqrt{3K}} (\alpha + B) \right\} = \left(\frac{\lambda - \sqrt{\lambda^2 - 12\mu}}{2\mu} \right)(G - r) .$$

Besides B, there is another constant, A, involved implicitly in this equation. We can, however, impose conditions say, $G(\infty) = G'(\infty) = 0$ so that, $A = 0$. In that case from (39) we have:

$$K\, G_\alpha^2 + \frac{1}{c_o^2}\, G^3 - \frac{G^2}{c_o}\, b = 0$$

which integrates into

$$Y_1 = \left(\frac{2}{3}\, c_o\, c_1 - \frac{c_o}{3}\sqrt{4c_1^2 - 6\pi_S}\right) + bc_o\, \operatorname{sech}^2 \frac{1}{2}\sqrt{bc_o}\left(\frac{\alpha}{c_o\sqrt{K}} + \frac{B_1}{2}\right)$$

where B_1 is again constant of integration giving us one parameter family of solitary waves. For some other set conditions at $\alpha = \infty$ for which $A \neq 0$, the solution is represented by (40) and represents the cnoidal waves. The equation (40) in terms of Y_1 can be written as

$$Y_1 = \left\{\frac{2}{3}\, c_1\, c_o - \frac{c_o}{3}\sqrt{4c_1^2 - 6\pi_S}\right\} + r + \frac{2\mu}{\lambda - \sqrt{\lambda^2 - 12\mu}} \quad x$$

(41)
$$\operatorname{cn}^2\left\{\frac{(\lambda^2 - 12\mu)^{1/4}}{4c_o\sqrt{3K}}\,(\alpha + B)\right\}$$

On the left hand side $\pi_S = 0$. And the resulting solution is given by:

(42) $$Y_1 = 2c_o\, c_1\, \operatorname{sech}^2 \frac{1}{2}\sqrt{2c_o\, c_1}\left(\frac{\alpha}{c_o\sqrt{K}} + B_2\right)$$

where B_2 is the constant of integration fixing the position of a particular wave.

Let $\mu > 0$

$$\alpha + B' = \pm 2 \int \frac{dT}{\sqrt{-\mu T^4 + \lambda T^2 - 3}}$$

For $\lambda \leq 0$, the above integral is imaginary. We, therefore, assume that $\lambda > 0$.

$$\alpha + B' = \pm 2 \int \frac{dT}{\sqrt{\mu(\alpha_1^2 - T^2)(T^2 - \beta_1^2}}$$

where $\alpha_1^2 = \frac{\lambda + \sqrt{\lambda^2 - 12\mu}}{2\mu}$ $\qquad \beta_1^2 = \frac{\lambda - \sqrt{\lambda^2 - 12\mu}}{2\mu}$

and $\lambda^2 \geq 12\mu$. These restrictions on λ lead to some restrictions on A, i.e. on the initial conditions.

$$\alpha + B' = \pm \frac{2}{\alpha_1\sqrt{\mu}} \, Sn^{-1} \frac{\sqrt{\alpha_1^2 - T^2}}{\sqrt{\alpha_1^2 - \beta_1^2}}$$

$$G - r = \frac{1}{\alpha_1^2 \, cn^2 \frac{\alpha_1\sqrt{\mu}}{2}(\alpha + B') + \beta_1^2 \, Sn^2 \frac{\alpha_1\sqrt{\mu}}{2}(\alpha + B')}$$

$$y_1 = r - \left\{ -\frac{2}{3} c_o c_1 + \frac{c_o}{3}\sqrt{4c_1^2 - 6\pi_S} \right\}$$

(43)

$$+ \frac{1}{\alpha_1^2 \, cn^2 \frac{\alpha_1\sqrt{\mu}}{2}(\alpha + B') + \beta_1^2 \, S_n^{\,2} \frac{\alpha_1\sqrt{\mu}}{2}(\alpha + B')}$$

which represents periodic waves of period 4K where

$$K = \int_0^1 \frac{dT}{\sqrt{(1 - T^2)(1 - \frac{\alpha_1^2 - \beta_1^2}{\alpha_1^2} T^2)}}$$

Finally, $\mu = 0$ implies $A = 0$, and the solution in this case has been found above as the solitary wave.

$$\mu = 3(3r^2 - 2rbc_o) = 3r(3r - 2bc_o)$$

$$\lambda = 3(-3r + bc_o)$$

For $\mu \leq 0$, $0 \leq r \leq \frac{2bc_o}{3}$

But if $r = \frac{2bc_o}{3}$, $\lambda < 0$ and the solution again is imaginary. Hence $0 \leq r < \frac{2bc_o}{3}$. The equation giving r is

$$(\frac{r}{bc_o})^3 - (\frac{r}{bc_o})^2 - \frac{A}{b^3 c_o} = 0$$

For $\frac{r}{bc_o} < \frac{2}{3}$, $A \leq 0$. Since the roots of a polynomial continuously depend upon its coefficients, A can be chosen, i.e. an appropriate set of initial conditions can be prescribed, such that at least one root of the above cubic will satisfy (44). For $\mu > 0$, we have either $r > \frac{2bc_o}{3}$ or $r < 0$. But in this case $\lambda > 0$, i.e. $r < \frac{bc_o}{3}$. The two together imply that $r < 0$. In this case we also have $\lambda^2 \geq 12\mu$, $\lambda^2 - 12\mu = 9(bc_o - r)(3r + bc_o) \geq 0$ if $3r + bc_o \geq 0$, i.e. $0 > r \geq -\frac{bc_o}{3}$. This again requires that $A \leq 0$.

The solutions in the two cases are given by the equations (41) and (43) respectively. Each one of them involve two

arbitrary constants. Being periodic solutions these are bounded over $0 \leq \alpha < \infty$. For $\pi_S > 0$ each has an elevation $< 2c_o c_1$ at $\alpha = 0$. This implies that the above constants can be evaluated if we impose two initial conditions at $\alpha = 0$ which are such that $y_1(0) < 2c_o c_1$. In particular if we choose them to satisfy the equation (42) we shall have a solution valid on the whole of the real line and smooth at least up to the first derivative at all points of it. Over $x \leq 0$ we shall have a solitary wave and it will be connected on $x \geq 0$ with another solitary wave or periodic wave depending on whether $A = 0$ or $A \neq 0$.

(ii) $K > 0$ $\qquad \pi_S = \begin{cases} \text{constant} & \text{for } |\alpha| \leq a \\ 0 & \text{for } |\alpha| > a \end{cases}$

We again refer back to the equation (38) with π_S as defined above. The solution on the segment is again by (41) and (43) which are two parameter families of periodic waves. On the half real line $-\infty < \alpha < a_1$, the solution is the one parameter family of solitary waves

$$y_1 = 2c_o c_1 \operatorname{sech}^2 \frac{1}{2}\sqrt{2c_o c_1}\left(\frac{\alpha}{c_o\sqrt{K}} + B_1\right) .$$

As discussed above these can be joined smoothly up to the first derivative. We call the above solutions W_1, W_2, respectively. On the segment $a < \alpha < \infty$, we have, as before,

two parameter family of waves which can be, likewise, connected with W_2 at $\alpha = a$.

CASE II: $K < 0 \Rightarrow (\eta_o - d)^2 < \frac{3T}{g\rho}$

i.e. depth of the water in equilibrium $< \sqrt{\frac{3T}{g\rho}}$. Apparently this case does not arise when $T = 0$. We put $K = -\ell$, $\ell > 0$ in the equation (38)

$$\ell Y_{1\alpha\alpha} - \frac{3}{2c_o^2} Y_1^2 + \frac{2c_1}{c_o} Y_1 = \pi_S$$

(i) $\pi_S = 0$ for $-\infty < \alpha < 0$

so that

$$\ell Y_{1\alpha\alpha} - \frac{3}{2c_o^2} \cdot Y_1^2 + \frac{2c_1}{c_o} Y_1 = 0$$

We write $Y_1 = -Z_1$. The above equation becomes

$$\ell Z_{1\alpha\alpha} + \frac{3}{2c_o^2} Z_1^2 + \frac{2c_1}{c_o} Z_1 = 0$$

This equation represents solitary wave only if $c_1 < 0$. Then it integrates into,

(46)
$$Z = -2c_1 c_o \operatorname{sech}^2 \frac{1}{2} \sqrt{-2c_1 c_o} \left(\frac{\alpha}{c_o\sqrt{\ell}} + d_1\right)$$

$$Y_1 = 2c_1 c_o \operatorname{sech}^2 \frac{1}{2} \sqrt{-2c_1 c_o} \left(\frac{\alpha}{c_o\sqrt{\ell}} + d_1\right)$$

Since $c_1 < 0$, this is a negative wave. d_1 is constant of integration which would represent a parameter.

On $0 < \alpha < \infty$, $\pi_S \neq 0$. The solution as before is

$$- Y_1 = - \frac{2}{3} c_1 c_o - \frac{c_o}{3} \sqrt{4c_1^2 - 6\pi_S} + r + \frac{2\mu}{\lambda - \sqrt{\lambda^2 - 12\mu}} x$$

$$cn^2 \frac{(\lambda^2 - 12\mu)^{1/4}}{4c_o \sqrt{3\ell}} (\alpha + B)$$

$$(47) \quad Y_1 = \frac{2}{3} c_1 c_o + \frac{c_o}{3} \sqrt{4c_1^2 - 6\pi_S} + r + \frac{2\mu}{\lambda - \sqrt{\lambda^2 - 12\mu}} x$$

$$cn^2 \left(\frac{(\lambda^2 - 12\mu)^{1/4}}{4c_o \sqrt{3\ell}} (\alpha + B) \right)$$

where λ, μ, r have the same meaning as before. Therefore the result in this case also is that the two waves can be joined smoothly at $\alpha = 0$. Similarly in (ii) when

$$\pi_S = \begin{cases} \text{constant} & \text{for } |\alpha| \leq a \\ 0 & \text{for } |\alpha| > a \end{cases}$$

We get the three solutions that can be connected smoothly up to the first derivative.

Finally, the complete solution consists of two or more solitary waves or the solitary waves and the periodic waves joined smoothly up to the first derivative at the points of discontinuities of the pressure. The levels at the two infinities need not be the same. The solitary waves can be positive or negative depending on whether the depth at $\alpha = -\infty$ is greater than or less than $\sqrt{\frac{3\Pi}{g\rho}}$.

Chapter III

Waves in Shallow Waters

Formulations of the problem

We consider water flowing in an infinite channel with finite depth; the bottom of the channel is considered horizontal. The assumptions on the flow and the water are the same as imposed analogously in Chapters I and II. We take the x-axis in the bottom plane and y-axis vertically upward. These axes are considered fixed in space. Let the free surface be given by $y = \eta(x,t)$. The Euler equations of motion are:

$$u_t + uu_x + vu_y = -\frac{1}{\rho} p_x \tag{1}$$

$$v_t + uv_x + vv_y = -\frac{1}{\rho} p_y - g \tag{2}$$

The equation of continuity is

$$u_x + v_y = 0 \tag{3}$$

Kinematical condition at the free surface is

$$\eta_t + u\eta_x - v = 0 \tag{4}$$

$$v = 0 \quad \text{at bottom.}$$

$$\pi - p_0 = \frac{-\tau\, \eta_{xx}}{(1+\eta_x^2)^{3/2}} \qquad y = \eta, \tag{5}$$

where p_0 is the atmospheric pressure at a point just outside the liquid and π is the pressure at the corresponding point just inside the liquid.

Integrating the equations of continuity:

$$v\Big|_0^\eta = -\int_0^\eta u_x \, dy$$

Substituting for v in (4):

$$\eta_t\Big|_0^\eta + u\eta_x\Big|_0^\eta = -\int_0^\eta u_x \, dy$$

Also we have:

$$\frac{\partial}{\partial x}\int_0^\eta u(x,y,t)\, dy = \int_0^\eta u_x \, dy + \eta_x u(x,y,t)\Big|_{y=\eta}$$

The two together =>

$$-\eta_t = \frac{\partial}{\partial x}\int_0^\eta u \, dy$$

For the shallow wave theory we have the hydrostatic law:

$$p - p_0 = -\frac{\tau\eta_{xx}}{(1+\eta_x^2)^{3/2}} + g\rho(\eta - y) \, .$$

It is obvious from this that p_x does not depend upon y. Hence if u is originally independent of y, it will be so always. We suppose it likewise so that $u_y = 0$.

The equations then reduce to

$$u_t + uu_x = -\frac{1}{\rho}\{g\rho\eta_x - \frac{\partial}{\partial x}\left(\frac{\tau\,\eta_{xx}}{(1+\eta_x^2)^{3/2}}\right)\}$$

i.e.

$$u_t + uu_x + g\eta_x = \frac{1}{\rho}\frac{\partial}{\partial x}\left(\frac{\tau\,\eta_{xx}}{(1+\eta_x^2)^{3/2}}\right) \tag{6}$$

$$-\eta_t = \frac{\partial}{\partial x}(u\eta) \tag{7}$$

If we suppose that water has zero surface tension, then it is well known that the equations (6), (7) with $\tau = 0$ has uniform flows as its only smooth solution. The problem dealt with here is the investigation of some other smooth flows when $\tau \neq 0$. In particular we look for progressive wave solutions.

Let

$$\eta = \eta(x - at) = \eta(\zeta)$$

$$u = u(x - at) = u(\zeta)$$

The above equations then become,

$$-au' + uu' + g\eta' = +\frac{\tau}{\rho}\frac{\partial}{\partial\zeta}\left(\frac{\eta_{\zeta\zeta}}{(1+\eta_x^2)^{3/2}}\right)$$

$$a\eta' = \frac{\partial}{\partial\zeta}(u\eta)$$

Integrating, we get

$$(8) \qquad \frac{1}{2}(u-a)^2 + g\eta = +\frac{\tau}{\rho}\frac{\eta_{\zeta\zeta}}{(1+\eta_\zeta^2)^{3/2}} + A$$

$$(9) \qquad a\eta = u\eta + B$$

We assume that at $\zeta = 0$, $\eta = h$, $\eta' = b$, $\eta'' = 0$, $u = u_0$, b is some positive constant. Then

$$A = \frac{1}{2}(u_0 - a)^2 + gh$$

$$B = (a - u_0)h \ .$$

Eliminating (u_0-a) from (8), (9) we get

$$\frac{\tau}{\rho}\frac{\eta''}{(1+\eta_\zeta^2)^{3/2}} + A = g\eta + \frac{1}{2}\frac{B^2}{\eta^2}$$

Let $\eta' = V$; the result is a first order ordinary differential equation:

$$\frac{\tau}{\rho}\frac{dV}{d\eta} = \frac{(1+V^2)^{3/2}}{V\eta^2}\left(g\eta^3 + \frac{1}{2}B^2 - A\eta^2\right)$$

$$(10) \qquad \frac{\tau}{\rho}\frac{dV}{d\eta} = \frac{(1+V^2)^{3/2}}{V\eta^2}(h-\eta)\{-g\eta^2 + \frac{1}{2}h(u_0-a)^2 + \frac{1}{2}\eta(u_0-a)^2\}.$$

This equation governs the disturbance about the following equilibrium positions which are the singular solutions: $V = 0$, $\eta = h$ and

$$\eta = \frac{-\frac{1}{2}(u_0-a)^2 \pm \{\frac{1}{4}(u_0-a)^4 + 2gh(u_0-a)^2\}^{1/2}}{-2g}$$

$$= \frac{(u_0-a)^2 \pm \{(u_0-a)^4 + 8gh(u_0-a)^2\}^{1/2}}{4g}$$

$$= \eta_+, \eta_- ,$$

where $\eta = \eta_+$ corresponds to the positive sign and

$\eta = \eta_-$ corresponds to negative sign.

The equilibrium positions, therefore, are

$$V = 0, \qquad \eta = h, \eta_+, \eta_-$$

To start with we consider the disturbance about $\eta = h$.

Set $\lambda = \eta - h$.

$$\frac{\tau}{\rho}\frac{dV}{d\lambda} = -\frac{(1+V^2)^{3/2}\lambda}{V(\lambda+h)^2}\{-g(\lambda+h)^2 + \frac{1}{2}h(u_0-a)^2 + \frac{1}{2}(u_0-a)^2(\lambda+h)\}$$

(11)

$$= -\frac{\{-1+(1+V^2)^{3/2}\}[-g\lambda^3+\lambda^2\{-2gh+\frac{1}{2}(u_0-a)^2\}+\lambda\{h(u_0-a)^2-gh^2\}]}{V(\lambda^2+2\lambda h+h^2)}$$

$$-\frac{[-g\lambda^3+\lambda^2\{-2gh+\frac{1}{2}(u_0-a)^2\}+\lambda\{h(u_0-a)^2-gh^2\}]}{V(\lambda^2+2\lambda h+h^2)}$$

from which it is obvious that in order to consider the behaviour of the solution about $\lambda = 0$ we have to study the equation:

$$\frac{\tau}{\rho}\ \frac{dV}{d\lambda} = -\ \lambda h\{(u_0-a)^2-gh\}/Vh^2.$$

We consider the two cases:

(a) $(u_0 -a)^2 \neq gh$

(b) $(u_0- a)^2 = gh$.

Case a. For $(u_0-a)^2 > gh$, $\eta = h$ is a center.

For $(u_0-a)^2 < gh$, $\eta = h$ is a saddle point.

For $(u_0-a)^2 < gh$,

$$\eta_+ = \frac{(u_0-a)^2 + \{(u_0-a)^4 + 8\ gh(u_0-a)^2\}^{1/2}}{4g}$$

$$< \frac{gh + (g^2h^2+8g^2h^2)^{1/2}}{4g} = h.$$

Therefore $\eta_+ < h$. Also $\eta_- < 0$. Hence $\eta_- < \eta_+ < h$.

For $(u_0-a)^2 > gh$,

$$\eta_+ = \frac{(u_0-a)^2 + \{(u_0-a)^4 + 8gh\}^{1/2}}{4g}$$

$$> \frac{gh + 3gh}{4g} = h.$$

In this case $\eta_- < h < \eta_+$.

We now consider the singularity at $\eta = \eta_+$.

$$\frac{\tau}{\rho}\ \frac{dV}{d\eta} = -\ \frac{(1+V^2)^{3/2}g}{V\ \eta^2}(h-\eta)(\eta-\eta_-)(\eta-\eta_+)$$

$$\eta - \eta_+ = \lambda\ .$$

Then

$$\frac{\tau}{\rho}\frac{dv}{d\lambda} = \frac{g(1+V^2)^{3/2}}{V(\lambda+\eta_+)^2}\,\lambda(\lambda+\eta_+-h)(\lambda+\eta_+-\eta_-)$$

$$= \frac{g\{(1+V^2)^{3/2}-1\}\{\lambda(\lambda+\lambda_+-h)(\lambda+\eta_+-\eta_-)\}}{V(\lambda+\eta_+)^2}$$

$$+\frac{g\lambda(\lambda+\eta_+-h)(\lambda+\eta_+-\eta_-)}{V(\lambda+\eta_+)^2}.$$

Whence it follows that we have to consider the linear equation:

$$(12)\qquad \frac{\tau}{\rho}\frac{dV}{d\lambda} = \frac{\lambda(\eta_+^2-\eta_+\eta_- - h\eta_+ + h\eta_-)}{V\eta_+^2}$$

implies $\eta = \eta_+$ is a saddle point for $(u_0-a)^2 > gh$

and $\eta = \eta_-$ is a center for $(u_0-a)^2 < gh$.

Similarly for $\eta = \eta_-$ we consider the linearized differential equation:

$$\frac{\tau}{\rho}\frac{dV}{d\lambda} = \frac{g\lambda(\eta_- - h)(\eta_- - \eta_+)}{V\eta_-^2}$$

i.e. $\eta = \eta_-$ is always a saddle point.

We know from Poincare's theory, that a center in the linearized differential equation goes into a center or spiral in the nonlinear case. It is, therefore, necessary to further analyze the differential equation (11) to determine the nature of its singularities which are analogous to centers in the corresponding linearized differential equation (12).

For this, we integrate (10),

$$\frac{\tau}{\rho}\ \frac{V}{(1+V^2)^{3/2}}\ \frac{dV}{d\eta} = g\eta + \frac{1}{2}\frac{B^2}{\eta^2} - A$$

$$-\frac{\tau}{\rho}\ \frac{1}{\sqrt{1+V^2}} = \frac{g\eta^2}{2} - \frac{1}{2}\frac{B^2}{\eta} - A\eta + \text{const.}$$

But $\zeta = 0$, $\eta = h$, $\eta' = b$, $u = u_0$. Therefore

$$-\frac{\tau}{\rho}\ \frac{1}{\sqrt{1+b^2}} = \frac{gh^2}{2} - \frac{1}{2}\frac{B^2}{h} - Ah + \text{const.}$$

$$\frac{\tau}{\rho}\left\{\frac{1}{\sqrt{1+b^2}} - \frac{1}{\sqrt{1+V^2}}\right\} = \frac{g}{2}(\eta^2-h^2) - \frac{1}{2}B^2\left(\frac{1}{\eta} - \frac{1}{h}\right) - A(\eta-h)$$

$$= \frac{1}{2}(\eta-h)\left[g(\eta+h) + \frac{h(u_0-a)^2}{\eta}\right.$$

$$\left. - (u_0-a)^2 - 2gh\right]$$

i.e.

$$\frac{\tau}{\rho}\left\{\frac{1}{\sqrt{1+b^2}} - \frac{1}{\sqrt{1+V^2}}\right\} = \frac{1}{2}\frac{(\eta-h)^2}{\eta}\left[g\eta - (u_0-a)^2\right] .$$

We write $gL = (u_0-a)^2 - gh$.

$$\frac{\tau}{\rho}\left\{\frac{1}{\sqrt{1+b^2}} - \frac{1}{\sqrt{1+V^2}}\right\} = \frac{1}{2}\frac{(\eta-h)^2}{\eta}\ (g\eta - gL - gh).$$

Let $\alpha = \rho g/2\tau$;

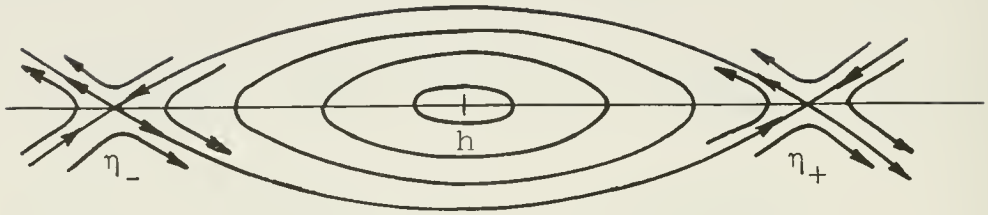
$$\frac{1}{\sqrt{1+V^2}} = \frac{1}{\sqrt{1+b^2}} - \frac{\alpha(\eta-h)^2}{\eta}\left\{(\eta-h)-L\right\} ;$$

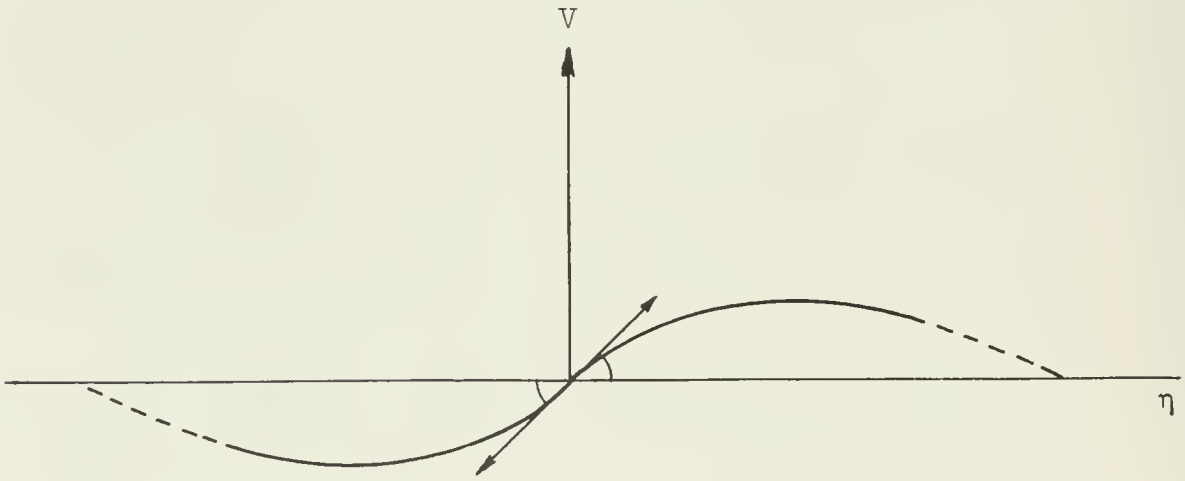
$$1 + V^2 = \frac{\eta^2}{\left[\frac{\eta}{\sqrt{1+b^2}} - \alpha(\eta-h)^2(\eta-h-L)\right]^2} .$$

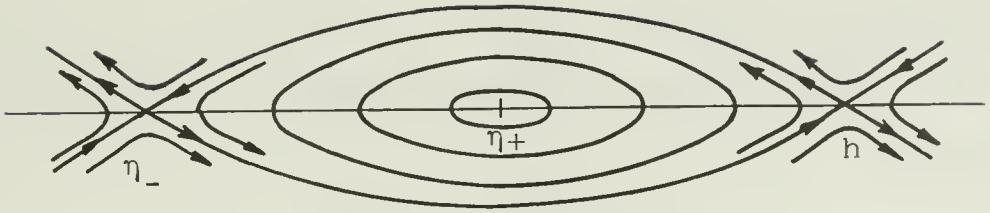
In order to see whether the singularities $\eta = h$, η_+ are centers or spirals, in the nonlinear case, we have only to take the intersection of this curve with a line passing through $\eta = h$, or $\eta = \eta_+$ as the case may be; i.e., with $V = m(\eta-h)$, $V = m(\eta-\eta_+)$ where m is a parameter which rotates the line through 2π.
Thus we have:

$$1 + m^2(\eta-h)^2 = \frac{\eta^2}{\left[\frac{\eta}{\sqrt{1+b^2}} - \alpha(\eta-h)^2(\eta-h-L)\right]^2}$$

which for any fixed m, is an eighth degree in η and so can't have more than eight roots. The spiral, on the other hand, must be intersected by any radial line in infinitely many points. Hence in both the cases the singularities at $\eta = h$, $\eta = \eta_+$ must be centers. See Figures 1 and 2 below. The saddle point shows that the disturbance initiated at $\eta = h$ reaches this level again at $\xi \longrightarrow \infty$ or $\xi \longrightarrow -\infty$, depending upon the sign of b. Thus in our case, it tends to form a crest on the right and a hollow on the left of the x-axis. In each case there is a single hump or hollow.


h
η_-
η_+


V
η


η+
η_
h

Case b. $(u_0 - a)^2 = gh$.

The differential equation in this case becomes:

$$\frac{\tau}{\rho}\frac{dV}{d\eta} = \frac{(1+V^2)^{3/2}}{V\eta^2}(h-\eta)(-g\eta^2+\frac{1}{2}gh^2+\frac{1}{2}gh\eta)$$

$$= \frac{g(1+V^2)^{3/2}}{2V\eta^2}(h-\eta)(-2\eta^2+h^2+h\eta)$$

$$= \frac{g(1+V^2)^{3/2}}{2V\eta^2}(h-\eta)(h-\eta)(2\eta+h)$$

$$(11)\quad \frac{dV}{d\eta} = \frac{\rho g}{2}\cdot\frac{(1+V^2)^{3/2}}{V\eta^2}(\eta-h)^2(2\eta+h).$$

The singular points are $V = 0$, $\eta = h$ and $V = 0$, $\eta = -h/2$. The former is a double singular point. Integrating (11), we get

$$1+V^2 = \frac{\eta^2}{\left[\frac{\eta}{\sqrt{1+b^2}} - \alpha(\eta-h)^3\right]^2}$$

where $\alpha = \rho g/2\tau$.

In order to discuss these curves in the vicinity of $\eta = h$, we have the substitution $\eta - h = \lambda$.

Then we have:

$$1 + V^2 = \frac{(\lambda+h)^2}{\left[\frac{\lambda+h}{\sqrt{1+b^2}} - \alpha\lambda^3\right]^2}$$

i.e.

$$1 + V^2 = \frac{1+b^2}{\left[1 - \frac{\alpha\sqrt{1+b^2}}{\lambda+h}\lambda^3\right]^2}$$

where λ is considered small.

This then in the vicinity of $\lambda = h$, yields:

$$1 + V^2 = (1+b^2)\left[1 + \frac{2\alpha\sqrt{1+b^2}}{h}\lambda^3\right]$$

$$V^2 = b^2 + \frac{2\alpha}{h}(1+b^2)^{3/2}\lambda^3$$

$$V^2 = \frac{2\alpha}{h}(1+b^2)^{3/2}\left\{\frac{hb^2}{2\alpha(1+b^2)^{3/2}} + \lambda^3\right\}$$

$$\frac{d\eta}{d\zeta} = \sqrt{\frac{2\alpha}{h}}(1+b^2)^{3/4}\left\{u^3 + \lambda^3\right\}^{1/2}$$

where $\mu^3 = \frac{hb^2}{2\alpha(1+b^2)^{3/2}}$. Let $B_1 = \sqrt{\frac{2\alpha}{h}}(1+b^2)^{3/4}$.

Then

$$\frac{d\eta}{d\zeta} = B_1\sqrt{u^3 + \lambda^3} = \frac{d\lambda}{d\zeta}$$

Integrating, we get

$$B_1\zeta + \text{const.} = \int^{\lambda} \frac{d\lambda}{\sqrt{u^3 + \lambda^3}}$$

We have to evaluate the integral:

$$I = \int \frac{d\lambda}{\sqrt{\lambda^3 + \mu^3}} = \int \frac{d\lambda}{\sqrt{(\lambda+\mu)}\ \sqrt{\lambda^2 - \lambda\mu + \mu^2}}$$

$$= -2\int \frac{dT}{\sqrt{3\mu T^4 + 1 - 3\mu T^2}}, \qquad \lambda+\mu = \frac{1}{T^2}$$

$$= -2\int \frac{dT}{[(T^2 - \sqrt{3r}T + r)(T^2 + \sqrt{3r}T + r)]^{1/2}}, \qquad r = \frac{1}{\sqrt{3\mu^2}}$$

$$= -\frac{4}{\sqrt{r}} \int \frac{dT_1}{\left\{\left(T_1^2 + \frac{2-\sqrt{3}}{2+\sqrt{3}}\right)\left(T_1^2 + \frac{2+\sqrt{3}}{2-\sqrt{3}}\right)\right\}^{1/2}}$$

$$T_1 = \frac{T+\sqrt{r}}{T-\sqrt{r}}$$

$$= -\frac{4}{\sqrt{r}} \int \frac{dT_1}{\{(T_1^2 + (2-\sqrt{3})^2)(T_1^2 + (2+\sqrt{3})^2)\}^{1/2}}$$

$$= \frac{4}{\sqrt{r}(2+\sqrt{3})} \int \frac{dS}{\sqrt{1-S^2}\ \sqrt{1 - \frac{8\sqrt{3}}{(2+\sqrt{3})^2} S^2}}$$

$$\text{for } (2+\sqrt{3})^2 + T_1^2 = \frac{(2+\sqrt{3})^2}{S^2}$$

$$B_1\xi + \text{const.} = \frac{4}{\sqrt{r}\ (2+\sqrt{3})} \mathrm{Sn}^{-1} S$$

$$= \frac{4}{\sqrt{r}\ (2+\sqrt{3})} \mathrm{Sn}^{-1} \frac{(2+\sqrt{3})(T-\sqrt{r})}{\{(2+\sqrt{3})^2(T-\sqrt{r})^2 + (T+\sqrt{r})^2\}^{1/2}}$$

$$\frac{(2+\sqrt{3})^2(T-\sqrt{r})^2}{(2+\sqrt{3})^2(T-\sqrt{r})^2+(T+\sqrt{r})^2} = Sn^2\ \frac{\sqrt{r}(2+\sqrt{3})}{4}\ (\beta_1\xi + d)\}$$

where d is an integration constant.

$$\frac{(T+\sqrt{r})^2}{(2+\sqrt{3})^2(T-\sqrt{r})^2} = \frac{cn^2\{\frac{\sqrt{r}}{4}(2+\sqrt{3})(\beta_1\xi + d)\}}{Sn^2\{\frac{\sqrt{r}}{4}(2+\sqrt{3})(\beta_1\xi + d)\}}$$

=>

$$r(\eta-h+\mu) = \left[\frac{(2+\sqrt{3})cn\{\frac{\sqrt{r}}{4}(2+\sqrt{3})(\beta_1\xi+d)\}-Sn\{\frac{\sqrt{r}}{4}(2+\sqrt{3})(\beta_1\xi+d)\}}{(2+\sqrt{3})cn\{\frac{\sqrt{r}}{4}(2+\sqrt{3})(\beta_1\xi+d)+Sn\{\frac{\sqrt{r}}{4}(2+\sqrt{3})(\beta_1\xi+d)\}}\right]^2$$

where $r = \frac{1}{\sqrt{3}\ \mu}$, $\mu^3 = \frac{hb^2}{2\alpha(1+h^2)^{3/2}}$, $\alpha = \frac{\rho g}{2\tau}$

$$\beta_1 = \sqrt{\frac{2\alpha}{h}}\ (1+ b^2)^{3/4}$$

=> that we have periodic waves of period

$$\frac{16(3^{1/4})\ b^{1/3}\ (1+b^2)^{1/2}\ h^{1/3}}{(2+\sqrt{3})(2\alpha)^{1/3}}\ K$$

where $K = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}}$, $k^2 = \frac{8\sqrt{3}}{(2+\sqrt{3})^2}$.

The other singularity is at V = 0, η = - b/2.
This is rejected in view of the fact that this is below the bottom of the channel.

Thus we have the following result:

For $\tau = 0$, we have uniform flows as the only smooth flows. When $\tau \neq 0$, we divide the solution in three parts. For $(u_0 - a)^2 < gh$, the disturbance starting at a point is carried over on both the sides of the source of disturbance. This disturbance propagates in a single crest on one side and in a single trough on the other side right up to $+\infty$ or $-\infty$ as the case may be. In case $(u_0 - a)^2 \geq gh$, the disturbance is propagated as a periodic wave.

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13. ABSTRACT

The purpose of the present work is to study surface waves in rectangular channels of finite depth. In all cases we consider two dimensional flow of an incompressible and nonviscous fluid. We also suppose that there is surface tension present in it.

In the first chapter we discuss the feasibility of a steady flow consistent with the linearized problem, and study the corresponding radiation conditions. Such a flow is found to be possible in both the supercritical and the subcritical cases but not in the critical case. In the supercritical case the disturbance dies both upstream and downstream; in the subcritical case, in general, it does not vanish downstream.

The second chapter deals with a study of solitary and periodic waves which result when a stream of uniform flow is disturbed by an arbitrary variable pressure on its free surface. We obtain a general differential equation of the motion. Explicit wave solutions are obtained by considering two special pressure distributions.

In the final chapter we study the progressive wave solutions of the zero order shallow water equations. In the subcritical case, we obtain a wave with a single hump and a single hollow on the two sides of the point of disturbance. In the critical and the supercritical cases, we obtain periodic waves.

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